

# Cubical Type Theory Inside a Presheaf Topos

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# Context

## Cubical Type Theory

- The first models of Homotopy Type Theory were non constructive.
- Thus, the need for Cubical Type Theory, based on cubical sets.
- Uses an interval object  $\mathbb{I}$ , which is a synthetic counterpart of  $[0, 1]$ .
- Leads to a notion of path  $p : \text{Path}_A a b$  is a map  $p : \mathbb{I} \rightarrow A$ .

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## What we will do

- It is well known that presheaves category have an internal type theory.
- We will try to understand what is needed from such a presheaves category to be a model of Cubical Type Theory.
- For that, we will use topos theory.

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# Cubical type theory

# Rules and interval

1. Standard Martin-Löf dependent type theory
2. An object  $\mathbb{I}$  which is the free de Morgan algebra on a fixed infinite set of names  $i, j, k, \dots$
3. Grammar of  $\mathbb{I}$  is

$$r, s ::= 0 \mid 1 \mid i \mid \neg r \mid r \wedge s \mid r \vee s$$

4. Custom  $\lambda$ -abstraction for  $i : \mathbb{I}$ :

$$\langle i \rangle . t$$

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$$\langle i \rangle . t$$

$\Gamma, \Delta$	$::=$	$() \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I}$	Contexts
$t, u, A, B$	$::=$	$x \mid \lambda x : A. t \mid t u \mid (x : A) \rightarrow B$	$\Pi$ -types
		$\mid (t, u) \mid t.1 \mid t.2 \mid (x : A) \times B$	$\Sigma$ -types
		$\mid \text{Path } A t u \mid \langle i \rangle t \mid t r$	Path types

# Interval

## Insight on $\mathbb{I}$

1.  $\mathbb{I}$  is a synthetic equivalent for  $[0, 1]$ .
2.  $\vee$  represents max
3.  $\wedge$  represents min
4.  $\neg$  represents  $1 - \cdot$ .
5. We write  $(i0)$  and  $(i1)$  for  $(i/0)$  and  $(i/1)$ .

## Jugdmental equality for $\mathbb{I}$

$\neg 0 = 1$	$\neg 1 = 0$	$\neg(r \vee s) = \neg r \wedge \neg s$	$\neg(r \wedge s) = \neg r \vee \neg s$
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# Path types

## Rules

$$\frac{\Gamma \vdash A \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A}{\Gamma \vdash \text{Path } A \ a \ b}$$

$$\frac{\Gamma \vdash p : \text{Path } A \ a \ b \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p \ r : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash p \ i = q \ i : A}{\Gamma \vdash p = q : \text{Path } A \ p_0 \ p_1}$$

$$\frac{\Gamma \vdash p : \text{Path } A \ p_0 \ p_1}{\Gamma \vdash p \ 0 = p_0 : A}$$

$$\frac{\Gamma \vdash p : \text{Path } A \ p_0 \ p_1}{\Gamma \vdash p \ 1 = p_1 : A}$$

$$\frac{\Gamma \vdash A \quad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle \ a : \text{Path } A \ a(i0) \ a(i1)}$$

$$\frac{\Gamma \vdash A \quad \Gamma, i : \mathbb{I} \vdash a : A \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle \ a) \ r = a(i/r) : A}$$

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$$\frac{\Gamma \vdash A \quad \Gamma, i : \mathbb{I} \vdash a : A \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle \ a) \ r = a(i/r) : A}$$

## Consequences

1. Reflexivity: For  $a : A$ ,  $1_a = \langle i \rangle \ a : \text{Path } A \ a \ a$
2. Function extensionality, from  $\Gamma \vdash p : (x : A) \rightarrow \text{Path } B \ (f \ x) \ (g \ x)$  we have

$$\Gamma \vdash \langle i \rangle \ \lambda x : A. p \ x \ i : \text{Path } ((x : A) \rightarrow B) \ f \ g$$

# Geometric intuition

In dimension  $n$

$n$  variables of dimension  $i_1, \dots, i_n : \mathbb{I}$  in the context, correspond to an  $n$ -dimensional cube.

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## In dimension two

$$i : \mathbb{I}, j : \mathbb{I} \vdash A$$

$$\begin{array}{ccc}
 A(i0)(j1) & \xrightarrow{A(j1)} & A(i1)(j1) \\
 \uparrow A(i0) & & \uparrow A(i1) \\
 A(i0)(j0) & \xrightarrow{A(j0)} & A(i1)(j0)
 \end{array}$$

# Face lattice

## Definition (Face lattice)

We define  $\mathbb{F}$  to be the distributive lattice generated by the symbols  $(i = 0)$  and  $(i = 1)$  (for all dimension name  $i$ ) with relation  $(i = 0) \wedge (i = 1) = 0_{\mathbb{F}}$ . The grammar is

$$\varphi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

We have the rule

$$\frac{\Gamma \vdash \psi : \mathbb{F}}{\Gamma, \psi \vdash}$$

## Contexts and congruence

For context  $\Gamma$ , we define recursively a congruence  $=_{\Gamma}$  on  $\mathbb{F}$ .

# Systems

## Definition (System)

A system is a compatible union of sub-polyhedra. The grammar is:

$$t, u, A, B ::= \dots \mid [\varphi_1 a_1 \dots \varphi_n a_n]$$

## Rules for systems

If  $\Gamma \vdash \varphi_1 \vee \dots \vee \varphi_n =_{\Gamma} 1_{\mathbb{F}} : \mathbb{F}$ , then

$$\frac{\Gamma, \varphi_1 \vdash a_1 : A \quad \dots \quad \Gamma, \varphi_n \vdash a_n : A \quad \Gamma, \varphi_i \wedge \varphi_j \vdash a_i = a_j : A \ (i \leq i, j \leq n)}{\Gamma \vdash [\varphi_1 a_1 \dots \varphi_n a_n] : A}$$

$$\frac{\Gamma, \varphi_1 \vdash J \quad \dots \quad \Gamma, \varphi_n \vdash J}{\Gamma \vdash J} \quad \frac{\Gamma \vdash [\varphi_1 a_1, \dots, \varphi_n a_n] : A \quad \Gamma \vdash \varphi_i = 1_{\mathbb{F}} : F}{\Gamma \vdash [\varphi_1 a_1, \dots, \varphi_n a_n] = a_i : A}$$

# Composition

## Notation

We write  $\Gamma \vdash a : A[\varphi \mapsto u]$  for  $\Gamma \vdash a : A$  and  $\Gamma, \varphi \vdash a = u : A$

## Definition (Composition)

Composition is a way to transport an element along a variable of dimension. The grammar is:

$$t, u, A, B ::= \dots \mid \text{comp}^i A [\varphi \mapsto u] a_0$$

with  $u$  a system. The rule is:

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \quad \Gamma, i : \mathbb{I} \vdash A \quad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \quad \Gamma \vdash a_0 : A(i_0)[\varphi \mapsto u(i_0)]}{\Gamma \vdash \text{comp}^i A [\varphi \mapsto u] a_0 : A(i_1)[\varphi \mapsto u(i_1)]}$$

# Transitivity

## Transitivity

Composition allows us to prove transitivity of paths. If  $\Gamma \vdash A$ ,  $\Gamma \vdash a, b, c : A$ , then

$$\frac{\Gamma \vdash p : \text{Path } A \ a \ b \quad \Gamma \vdash q : \text{Path } A \ b \ c}{\Gamma \vdash \langle i \rangle \text{ comp}^j A [(i = 0) \vee (i = 1) \mapsto [(i = 0) a, (i = 1) (q j)]] (p i) : \text{Path } A \ a \ c}$$



# Contractible types and equivalences

## Definition (Contractible types)

A type  $A$  is contractible if

$$\text{isContr } A \triangleq (x : A) \times ((y : A) \rightarrow \text{Path } A \ x \ y)$$

is inhabited.

# Contractible types and equivalences

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is inhabited.

## Definition (Equivalence)

Given two types  $T, A$  and  $f : T \rightarrow A$ , we define

$$\text{isEquiv } T \ A \ f \triangleq (y : A) \rightarrow \text{isContr } ((x : T) \times \text{Path } A \ y \ (f \ x))$$

We define the type

$$\text{Equiv } T \ A \triangleq (f : T \rightarrow A) \times \text{isEquiv } T \ A \ f$$

# Glueing

## Definition (Glueing)

The glueing operation is the analogous of composition, but for types. The formation rule is:

$$\frac{\Gamma \vdash A \quad \Gamma, \varphi \vdash T \quad \Gamma, \varphi \vdash f : \text{Equiv } T \ A}{\Gamma \vdash \text{Glue } [\varphi \mapsto (T, f)] A}$$

# Glueing

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## Intuition for glueing

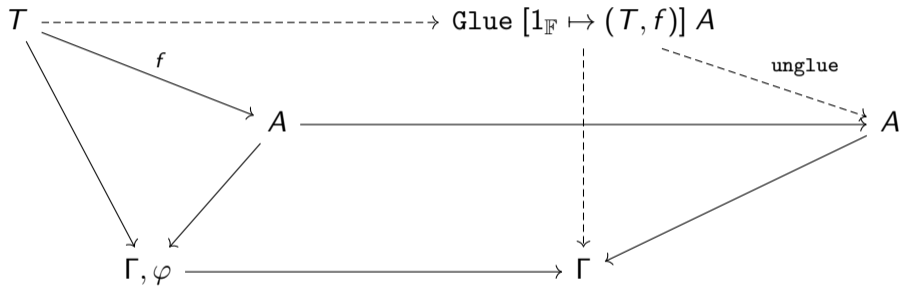
If

1.  $\Gamma \vdash A$
2.  $\Gamma, \varphi \vdash T$
3.  $A$  and  $T$  are equivalent on the region  $\varphi$

then we have the equality  $\Gamma, \varphi \vdash \text{Glue } [\varphi \mapsto (T, f)] A = T$ .

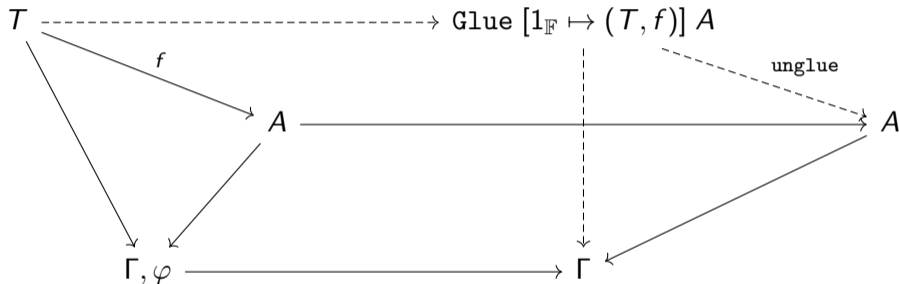
# Glueing and univalence

We can illustrate the type  $\text{Glue} [1_{\mathbb{F}} \mapsto (T, f)] A$  as:



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## Theorem (Univalence in Cubical Type Theory)

For any term

$$t : (A B : U) \rightarrow \text{Path } U A B \rightarrow \text{Equiv } A B$$

the map  $t A B : \text{Path } U A B \rightarrow \text{Equiv } A B$  is an equivalence.

# Logic inside a presheaf topos

# Topos

## Definition (A definition of a topos)

A *topos* is a category that

- has finite limits
- is cartesian closed
- has a subobject classifier  $(\Omega, \text{true} : \mathbf{1} \rightarrow \Omega)$



# Topos

## Definition (A definition of a topos)

A *topos* is a category that

- has finite limits
- is cartesian closed
- has a subobject classifier  $(\Omega, \text{true} : \mathbf{1} \rightarrow \Omega)$

## Definition (Subobject classifier)

For any mono  $s : S \rightarrow X$ , there exists a unique  $\chi_s : X \rightarrow \Omega$  such that the following is a pullback:

$$\begin{array}{ccc}
 S & \longrightarrow & \mathbf{1} \\
 s \downarrow & & \downarrow \text{true} \\
 X & \xrightarrow{\chi_s} & \Omega
 \end{array}$$

# Presheaf topos

## Condition for the classifier

For any  $X \in \mathcal{E}$ , we have

$$\text{Sub}(X) \simeq \mathcal{E}(X, \Omega)$$

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## Presheaf topos

Let  $\mathcal{C}$  be a locally small category, then  $\mathcal{E} = [\mathcal{C}^{\text{op}}, \text{Sets}]$  is a topos. The subobject classifier  $\Omega \in \mathcal{E}$  maps each  $c \in \mathcal{C}$  to the set of sieves on  $c$ . We define  $\text{true} : \mathbf{1} \rightarrow \Omega$  by picking the maximal sieve.

# Kripke-Joyal semantics

## Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.

# Kripke-Joyal semantics

## Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.
- Then, we use the notion of Kripke-Joyal forcing to recursively unwind formulas,
- Thus transforming a formula into a concrete object of the topos.

# Heyting structure

## Definition (Heyting algebra)

A *Heyting algebra* is a bounded lattice with an operation  $\Rightarrow$  such that

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## Theorem (Heyting algebra of the subobjects)

For any  $X \in \mathcal{E}$ ,  $\text{Sub}(X)$  is a Heyting algebra.

# Top and Bot

 $\top$ 

We take  $\top := \text{true}$ . It is the classifying map of  $! : \mathbf{1} \rightarrow \mathbf{1}$ . We have the following pullback.

$$\begin{array}{ccc}
 \mathbf{1} & \longrightarrow & \mathbf{1} \\
 \downarrow ! & & \downarrow \\
 \mathbf{1} & \xrightarrow{\top} & \Omega
 \end{array}$$



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## $\perp$

We define  $\perp$  with the following pullback.

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathbf{1} \\ \downarrow ! & & \downarrow \text{true} \\ \mathbf{1} & \xrightarrow{\perp} & \Omega \end{array}$$

## And

 $\wedge$ 

We have  $\wedge : \Omega \times \Omega \rightarrow \Omega$  with the following pullback.

$$\begin{array}{ccc}
 \mathbf{1} & \longrightarrow & \mathbf{1} \\
 \langle \text{true}, \text{true} \rangle \downarrow & & \downarrow \text{true} \\
 \Omega \times \Omega & \xrightarrow{\wedge} & \Omega
 \end{array}$$

# Image

## Definition (Image)

A mono  $m$  is the image of an arrow  $f$ , if  $f = m \circ e$  for some epi  $e$  and if whenever  $f$  factors through a mono  $m'$  then  $m$  factor through  $m'$ . We write  $m = \text{im}(f)$ .

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## Theorem (Image in a topos)

Let  $f : X \rightarrow Y$ . Then  $f$  has an image  $m : M \rightarrow Y$  and factors as in

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow e & \nearrow m \\ & M & \end{array}$$

with  $e$  epi.

Or

Setup for the  $\vee$ 

$$\begin{array}{ccccc}
 \Omega & \xrightarrow{l_1} & \Omega + \Omega & \xleftarrow{l_2} & \Omega \\
 & \searrow \langle \text{id}, \text{true} \rangle & \vdots & \swarrow \langle \text{true}, \text{id} \rangle & \\
 & & U & & \\
 & & \vdots & & \\
 & & u & & \\
 & \searrow & \Omega \times \Omega & \swarrow & \\
 & & & & 
 \end{array}$$

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 \langle \text{id}, \text{true} \rangle & & \downarrow u & & \langle \text{true}, \text{id} \rangle \\
 & \searrow & \Omega \times \Omega & \swarrow & \\
 & & & & 
 \end{array}$$

 $\vee$ 

We take  $\vee : \Omega \times \Omega \rightarrow \Omega$  to be the classifying arrow of  $u : U \rightarrow \Omega \times \Omega$  above.

# Quantifiers as pullbacks

## Adjonction to the pullback functor

We define

$$\begin{array}{ccc}
 \mathcal{E}(Y \times X, \Omega) \simeq \text{Sub}(Y \times X) & & \\
 \begin{array}{c} \curvearrowright \\ \exists_Y \\ \downarrow \\ \mathcal{E}(X, \Omega) \simeq \text{Sub}(X) \end{array} & \begin{array}{c} \uparrow \pi^* \\ \downarrow \\ \mathcal{E}(X, \Omega) \simeq \text{Sub}(X) \end{array} & \begin{array}{c} \curvearrowleft \\ \forall_Y \\ \downarrow \\ \mathcal{E}(X, \Omega) \simeq \text{Sub}(X) \end{array}
 \end{array}$$

where  $\pi^*$  is the pullback functor along the second projection  $\pi : Y \times X \rightarrow X$  and

$$\exists_Y \dashv \pi^* \dashv \forall_Y$$

# Interpretation of formulas

## Interpretation of formulas

Suppose we have  $\sigma, \tau : X \rightarrow \Omega$ ,  $\theta : Y \times X \rightarrow \Omega$ , then we interpret

- $\sigma \wedge \tau$  by  $\wedge \circ \langle \sigma, \tau \rangle : X \rightarrow \Omega$
- $\sigma \vee \tau$  by  $\vee \circ \langle \sigma, \tau \rangle : X \rightarrow \Omega$
- $\sigma \Rightarrow \tau$  by  $\Rightarrow \circ \langle \sigma, \tau \rangle : X \rightarrow \Omega$
- $\neg \sigma$  by  $\Rightarrow \circ \langle \sigma, \perp \circ ! \rangle = \neg \circ \sigma : X \rightarrow \Omega$
- $\forall y : Y, \theta(y)$  by  $\forall_Y \theta : X \rightarrow \Omega$
- $\exists y : Y, \theta(y)$  by  $\exists_Y \theta : X \rightarrow \Omega$



# A very important notation

## Notation

Every  $\sigma : X \rightarrow \Omega$  is to be thought as a formula. Thus we write

$$\{x : X \mid \sigma(x)\}$$

for the subobject of  $X$  arising from the pullback of  $\sigma$  and true.

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$$\begin{array}{ccc}
 \{x : X \mid \sigma(x)\} & \longrightarrow & \mathbf{1} \\
 \downarrow & & \downarrow \text{true} \\
 X & \xrightarrow{\sigma} & \Omega
 \end{array}$$

# Interpretation of equality

## Interpretation of =

For all  $Y \in \mathcal{E}$ , let  $\Delta_Y : Y \rightarrow Y \times Y$  be the diagonal map and let  $\delta_Y$  be its classifying arrow. Let  $f, g : X \rightarrow Y$ . We interpret  $f = g$  by  $\delta_Y \circ \langle f, g \rangle$ . Thus we have the following pullback.

$$\begin{array}{ccc}
 \{x : X \mid f(x) = g(x)\} & \longrightarrow & \mathbf{1} \\
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 X & \xrightarrow{f=g} & \Omega
 \end{array}$$

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 \end{array}$$

## Theorem

*The following diagram is an equalizer.*

$$\{x : X \mid f(x) = g(x)\} \twoheadrightarrow X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$$

# Validity of a formula

## Definition (Validity)

Let  $\sigma : X \rightarrow \Omega$ . We say that  $\sigma$  is *valid* whenever  $\sigma$  factors through  $\text{true} : \mathbf{1} \rightarrow \Omega$ .

$$\begin{array}{ccc}
 \mathbf{1} & & \\
 \uparrow & \searrow \text{true} & \\
 ! & & \Omega \\
 X & \xrightarrow{\sigma} & 
 \end{array}$$

In that case, we write  $X \vdash \sigma$ . If  $\sigma$  is a closed formula, then we write  $\vdash \sigma$  and this amounts to say that  $\sigma = \text{true}$ .

# Characterization of validity

Let  $X$  be a type and  $\sigma : X \rightarrow \Omega$ . The following are equivalent.

1.  $X \vdash \sigma$
2. The subobject  $\{x : X \mid \sigma(x)\} \multimap X$  admits a section
3.  $\{x : X \mid \sigma(x)\} \simeq X$

# Kripke-Joyal forcing

## Definition (Forcing)

Let  $\sigma : X \rightarrow \Omega$  be a formula and  $x : \mathbf{y}c \rightarrow X$ . We say that  $x$  forces  $\sigma$  at stage  $c$ , written  $c \Vdash \sigma(x)$ , if the following dotted arrow exists, making the left triangle commute.

$$\begin{array}{ccccc}
 & & \{x : X \mid \sigma\} & \longrightarrow & \mathbf{1} \\
 & \nearrow \text{dotted} & \downarrow s & & \downarrow \text{true} \\
 \mathbf{y}c & \xrightarrow{x} & X & \xrightarrow{\sigma} & \Omega
 \end{array}$$

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 \end{array}$$

## Theorem

Let  $\sigma : X \rightarrow \Omega$ .  $X \Vdash \sigma$  if and only if  $c \Vdash \sigma(x)$  for all  $x : \mathbf{y}c \rightarrow X$ .



# More theorems

## Theorem (Monotonicity)

*Let  $\sigma : X \rightarrow \Omega$  be a formula and  $x : \mathbf{y}c \rightarrow X$  such that  $c \Vdash \sigma(x)$ , then for all  $f : b \rightarrow c$ ,  $b \Vdash \sigma(xf)$ .*

# More theorems

## Theorem (Monotonicity)

Let  $\sigma : X \rightarrow \Omega$  be a formula and  $x : \mathbf{y}c \rightarrow X$  such that  $c \Vdash \sigma(x)$ , then for all  $f : b \rightarrow c$ ,  $b \Vdash \sigma(xf)$ .

## Theorem (Forcing with a terminal object)

If  $\mathcal{C}$  has a terminal object  $t \in \mathcal{C}$ , then a closed formula  $\sigma : \mathbf{1} \rightarrow \Omega$  is valid if and only if  $t \Vdash \sigma$ .

# The main theorem

## Theorem (Conditions for forcing)

Let  $\sigma, \tau : X \rightarrow \Omega$ ,  $\theta : Y \times X \rightarrow \Omega$  and  $x : \mathbf{y}c \rightarrow X$ , then

- $c \Vdash \perp$  *never*
- $c \Vdash \top$  *always*
- $c \Vdash \sigma(x) \wedge \tau(x)$  *if and only if*  $c \Vdash \sigma(x)$  *and*  $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \vee \tau(x)$  *if and only if*  $c \Vdash \sigma(x)$  *or*  $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \Rightarrow \tau(x)$  *if and only if* *for all*  $f : d \rightarrow c$ ,  $d \Vdash \sigma(xf)$  *implies*  $d \Vdash \tau(xf)$
- $c \Vdash \neg \sigma(x)$  *if and only if* *for all*  $f : d \rightarrow c$ , *we do not have*  $d \Vdash \sigma(xf)$
- $c \Vdash \exists y : Y, \theta(y, x)$  *if and only if*  $c \Vdash \theta(y, x)$  *for some*  $y : \mathbf{y}c \rightarrow Y$
- $c \Vdash \forall y : Y, \theta(y, x)$  *if and only if*  $d \Vdash \theta(y, xf)$  *for all*  $f : d \rightarrow c$  *and*  $y : \mathbf{y}d \rightarrow Y$

# The main theorem

## Theorem (Conditions for forcing)

Let  $\sigma, \tau : X \rightarrow \Omega$ ,  $\theta : Y \times X \rightarrow \Omega$  and  $x : \mathbf{y}c \rightarrow X$ , then

- $c \Vdash \perp$  *never*
- $c \Vdash \top$  *always*
- $c \Vdash \sigma(x) \wedge \tau(x)$  *if and only if*  $c \Vdash \sigma(x)$  *and*  $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \vee \tau(x)$  *if and only if*  $c \Vdash \sigma(x)$  *or*  $c \Vdash \tau(x)$
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- $c \Vdash \forall y : Y, \theta(y, x)$  *if and only if*  $d \Vdash \theta(y, xf)$  *for all*  $f : d \rightarrow c$  *and*  $y : \mathbf{y}d \rightarrow Y$

We can recursively unwind the condition  $c \Vdash \sigma(x)$

# Forcing for equality

## Theorem (Forcing for equality)

Let  $f, g : X \rightarrow Y$  and  $x : \mathbf{y}c \rightarrow X$ , then  $c \Vdash f(x) = g(x)$  if and only if  $f \circ x = g \circ x$  as maps of  $\mathcal{E}$ .

The proof uses the equalizer characterization of equality:

$$\begin{array}{ccccc}
 \{x : X \mid f(x) = g(x)\} & \longrightarrow & X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y \\
 & & \uparrow x & & \\
 & & \mathbf{y}c & & 
 \end{array}$$

A dotted arrow points from  $\mathbf{y}c$  to  $\{x : X \mid f(x) = g(x)\}$ .

# Martin-Löf Type Theory inside a presheaf category

# Overview

We will follow [AGH21] to define small maps, and the small map classifier  $\pi$  given by a Hofmann-Streicher lifting of a Grothendieck universe.

- We fix  $\kappa$  a (strongly) inaccessible cardinal.
- A set *small* if it has cardinality less than  $\kappa$ .
- We write  $\text{Sets}_\kappa$  for the full subcategory of  $\text{Sets}$  consisting of small sets (which is a Grothendieck universe).
- We fix a small (in the above sense) category  $\mathcal{C}$ .
- We call  $\mathcal{E}$  the associated presheaf topos.

# Hofmann-Streicher lifting

## Definition (Small maps)

1. We say that a presheaf  $A \in \mathcal{E}$  is *small* if  $A(c)$  is a small set, for all  $c \in \mathcal{C}$
2. We say that  $p : A \rightarrow X$  in  $\mathcal{E}$  is a *small map* if, for every  $x : \mathbf{y}c \rightarrow X$ , the presheaf  $A_x$  obtained by the pullback

$$\begin{array}{ccc}
 A_x & \longrightarrow & A \\
 \downarrow & & \downarrow p \\
 \mathbf{y}c & \xrightarrow{x} & X
 \end{array}$$

is small.



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is small.

## Class of small maps

We call  $\mathcal{S}$  the class of small maps in  $\mathcal{E}$ . In the same way that we can classify the monos  $S \rightarrow X$  with the map  $\text{true} : \mathbf{1} \rightarrow \Omega$ , we define  $\pi : E \rightarrow U$  that classifies the maps of  $\mathcal{S}$ .

# Classification

Given a small map  $p : A \rightarrow X$ , there exists a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ p \downarrow & & \downarrow \pi \\ X & \xrightarrow{c_p} & U \end{array}$$

We say that  $p$  is *classified* by  $c_p$ . Conversely, we introduce a canonical pullback  $p_A$  for each  $A : X \rightarrow U$ :

$$\begin{array}{ccc} X.A & \longrightarrow & E \\ p_A \downarrow & & \downarrow \pi \\ X & \xrightarrow{A} & U \end{array}$$

The map  $p_A$  is called the *display map* of  $A$ .

# Small maps and subobject classifier

## Theorem

For all  $X \in \mathcal{E}$ , we have

$$\mathcal{E}(X, U) \simeq \{u : (f X)^{\text{op}} \rightarrow \mathcal{S}\}$$

## Similarity with $\Omega$

We call  $\text{sf}(X) = \{u : (f X)^{\text{op}} \rightarrow \mathcal{S}\}$  the small families over  $X$ , then we have

$$\text{sf}(X) \simeq \mathcal{E}(X, U)$$

and with the subobject classifier, we have

$$\text{Sub}(X) \simeq \mathcal{E}(X, \Omega)$$

# $\Omega$ is a type of proposition

There is an inclusion maps  $\{-\} : \Omega \rightarrow U$  which is a monomorphism and fits into a pullback of the form:

$$\begin{array}{ccc}
 \mathbf{1} & \longrightarrow & E \\
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Conversely, by considering the image of the universal map  $\text{im}(\pi) \twoheadrightarrow U$ , and  $\{\text{supp}(-)\} : U \rightarrow \Omega$  is the propositional truncation.

$$\begin{array}{ccc}
 \text{im}(\pi) & \longrightarrow & \mathbf{1} \\
 \downarrow & & \downarrow \text{true} \\
 U & \xrightarrow{\text{supp}} & \Omega
 \end{array}$$

# Category with families

## Definition (Category with families)

The *category with families* associated to any presheaf topos  $\mathcal{E}$ , is defined as follows:

- The contexts are the objects  $X \in \mathcal{E}$
- A type  $A$  in context  $X$  is a map  $A : X \rightarrow U$
- A term  $a : A$  in context  $X$  is a map  $a : X \rightarrow E$  such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{a} & E \\ \parallel & & \downarrow \pi \\ X & \xrightarrow{A} & U \end{array}$$

- Definitional equality on terms or type is defined via equality of maps in the topos. For instance, we have  $X \vdash A = B$  if and only if  $A : X \rightarrow U$  and  $B : X \rightarrow U$  are the same maps in  $\mathcal{E}$ . For terms, we write  $X \vdash a = b : A$ .

# Some results

Theorem (Closed types are presheaves)

*Closed types  $A : \mathbf{1} \rightarrow U$  correspond, up to isomorphism, to small presheaves.*

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*Closed types  $A : \mathbf{1} \rightarrow U$  correspond, up to isomorphism, to small presheaves.*

## Theorem (Display maps and sections)

*Let  $A : \Gamma \rightarrow U$  be a type in context  $\Gamma$ . The elements  $\Gamma \vdash a : A$  are in bijective correspondence with the sections of the display map  $p_A : \Gamma.A \rightarrow \Gamma$ .*



# Type theory of a category with family

## Substitution

If  $\Delta, \Gamma$  are two contexts, and  $A : \Gamma \rightarrow U$  a type in context  $\Gamma$ , then we define the substitution  $A[t] = A \circ t$ , which allows us to have the following substitution rule valid in our type theory.

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash b(a) : B(a)}$$

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## Internal type theory

It is possible to internalize every basic construction of dependent type theory, and their formation, introduction, elimination, computation, and expansion rules [Awo18].

# Extending the forcing

## Definition

Let  $A : X \rightarrow U$  be a type in context  $X$ , and  $x : \mathbf{y}c \rightarrow X$ . For  $a : \mathbf{y}c \rightarrow E$ , We say  $c$  forces  $a : A(x)$  written  $c \Vdash a : A(x)$  if the following diagram commutes.

$$\begin{array}{ccc} \mathbf{y}c & \xrightarrow{a} & E \\ x \downarrow & & \downarrow \pi \\ X & \xrightarrow{A} & U \end{array}$$

Like in the standard forcing,  $c \Vdash a : A(x)$  is to say  $\mathbf{y}c \vdash a : A(x)$ :

$$\begin{array}{ccccc} \mathbf{y}c & \xrightarrow{a} & & & E \\ \parallel & & & & \downarrow \pi \\ \mathbf{y}c & \xrightarrow{x} & X & \xrightarrow{A} & U \end{array}$$

# Theorem for the forcing

## Theorem

Let  $A : X \rightarrow U$  be a type in context  $X$ , and  $x : \mathbf{y}c \rightarrow X$ . An element  $a : \mathbf{y}c \rightarrow E$  is the same thing as the dotted arrow in the following diagram:

$$\begin{array}{ccccc}
 & & X.A & \xrightarrow{q_A} & E \\
 & \nearrow u & \downarrow p_A & & \downarrow \pi \\
 \mathbf{y}c & \xrightarrow{x} & X & \xrightarrow{A} & U
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## Theorem

The data of  $a : X \rightarrow E$  such that  $X \vdash a : A$  is the same as families of elements  $a_x : \mathbf{y}c \rightarrow E$  such that  $c \Vdash a_x : A(x)$ , and are uniform in the sense that  $c \Vdash a_x = a(x) : A(x)$ .

# Main theorem

## Theorem (Condition for forcing)

1.  $c \Vdash a : 0$  never
2.  $c \Vdash a : 1$  for a unique  $a = \star : \mathbf{y}c \rightarrow E$
3.  $c \Vdash t : (A \times B)(x)$  if and only if  $c \Vdash \pi_1(t) : A(x)$  and  $c \Vdash \pi_2(t) : B(x)$
4.  $c \Vdash t : (A + B)(x)$  if and only if  $c \Vdash a : A(x)$  with  $t = \text{inl}(a)$  or  $c \Vdash b : B(x)$  with  $t = \text{inr}(a)$
5. We also have conditions for dependent sums and products.

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5. We also have conditions for dependent sums and products.

## Theorem (Relation the old forcing)

Let  $\sigma : X \rightarrow \Omega$  be a proposition and  $x : \mathbf{y}c \rightarrow X$ . Then the following are equivalent.

- (i)  $c \Vdash_{std} \sigma(x)$
- (ii)  $c \Vdash s : \sigma(x)$  for a (necessarily unique)  $s : \mathbf{y}c \rightarrow E$

# A model of cubical type theory



# Cubical presheaves

- We introduce the category  $\square$ . It will be the base category of a presheaf topos whose internal type theory will model cubical type theory.
- The category  $\square$  is the one from [CCHM15].
- Its objects are free de Morgan algebras, whose structure will be helpful to obtain an object  $\mathbb{I}$ .
- Then, we introduce the notion of cofibration, whose behavior is important to internalize Kan filling and glueing [OP16].

# The box category

For  $n \geq 0$ , we denote by  $I_n$  the free de Morgan algebra on  $n$  generators.

## Definition ( $\square$ )

We call  $\square$  the category having as objects cardinal numbers  $[n] \geq 0$  and as morphisms in  $\square([n], [m])$  the de Morgan homomorphisms  $f : I_m \rightarrow I_n$ .

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1. We specialize the previous sections to the case where  $\mathcal{C} = \square$ , so  $\mathcal{E}$  is the presheaf topos  $[\square^{\text{op}}, \text{Sets}]$ .
2.  $X_n \triangleq X([n])$ .
3.  $\mathbb{I} \triangleq \mathbf{y}[1]$ .
4.  $\mathbb{I}_n \triangleq \square([n], [1])$  has a de Morgan structure defined pointwise.
5. We call  $\sqcap_n, \sqcup_n, 0_n, 1_n$  the product, sum, zero, and one of this de Morgan algebra (that we can collect into natural transformations)

# Results about $\square$

## Theorem (Behavior of $\mathbb{I}$ )

- (i)  $0 \neq 1$
- (ii)  $\forall x : \mathbf{y}[n] \rightarrow \mathbb{I}$ , we have  $0 \sqcap x = 0 = x \sqcap 0$  and  $1 \sqcap x = x = x \sqcap 1$
- (iii)  $\forall x : \mathbf{y}[n] \rightarrow \mathbb{I}$ , we have  $0 \sqcup x = x = x \sqcup 0$  and  $1 \sqcup x = 1 = x \sqcup 1$

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## Theorem

For all  $[n] \in \square$ ,  $\mathbb{I}_n$  has decidable equality.

# Cofibrations

## Idea

Cofibrations are a way to internalize composition and filling introduced in [CCHM15]. We assume a map  $\text{cof} : \Omega \rightarrow \Omega$  and we consider the associated subobject

$$\text{Cof} = \{\varphi : \Omega \mid \text{cof } \varphi\}$$

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A *cofibration* is a monomorphism whose classifying arrow factors through  $\text{Cof} \rightarrow \Omega$ .



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A *cofibration* is a monomorphism whose classifying arrow factors through  $\text{Cof} \rightarrow \Omega$ .

## Theorem

Let  $\varphi : X \rightarrow \Omega$  be a proposition. For every  $x : \mathbf{y}c \rightarrow X$ , the following are equivalent.

1.  $c \Vdash \text{cof } \varphi(x)$
2.  $\varphi \circ x : \mathbf{y}c \rightarrow \Omega$  factor through  $\text{Cof} \rightarrow \Omega$

# Axiom to model CTT

Consider the following nine axioms:

$$\mathbf{ax}_1 \quad \forall \varphi : \mathbb{I} \rightarrow \Omega, (\forall i : \mathbb{I}, \varphi i \vee \neg \varphi i) \Rightarrow (\forall i : \mathbb{I}, \varphi i) \vee (\forall i : \mathbb{I}, \neg \varphi i)$$

$$\mathbf{ax}_2 \quad \neg(0 = 1)$$

$$\mathbf{ax}_3 \quad \forall i : \mathbb{I}, 0 \sqcap i = 0 = i \sqcap 0 \wedge 1 \sqcap i = i = i \sqcap 1$$

$$\mathbf{ax}_4 \quad \forall i : \mathbb{I}, 0 \sqcup i = i = i \sqcup 0 \wedge 1 \sqcup i = 1 = i \sqcup 1$$

$$\mathbf{ax}_5 \quad \forall i : \mathbb{I}, \text{cof}(i = 0) \wedge \text{cof}(i = 1)$$

$$\mathbf{ax}_6 \quad \forall \varphi \psi : \Omega, \text{cof } \varphi \Rightarrow \text{cof } \psi \Rightarrow \text{cof}(\varphi \vee \psi)$$

$$\mathbf{ax}_7 \quad \forall \varphi \psi : \Omega, \text{cof } \varphi \Rightarrow (\varphi \Rightarrow \text{cof } \psi) \Rightarrow \text{cof}(\varphi \wedge \psi)$$

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$$\mathbf{ax}_9 \quad (\varphi : \text{Cof})(A : [\varphi] \rightarrow U)(B : U)(s : (u : [\varphi]) \rightarrow (A u \simeq B)) \rightarrow (B' : U) \times \{s' : B' \simeq B \mid \forall u : [\varphi], A u = B' \wedge s u = s'\}$$

# Axioms to model CTT

## Theorem ([OP16])

*The category with families of a presheaf topos is a model of cubical type theory if its internal logic satisfies the nine axioms above.*

# The axioms, detailed

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# Kripke-Joyal forcing to prove the axioms

1. To prove each of these axioms, we can use Kripke-Joyal forcing and unravel each formula.
2. Since  $\square$  has a terminal object  $[0]$ , it suffices to prove that each axiom is forced at stage  $[0]$ . That is, for  $\mathbf{k} = \mathbf{1}, \dots, \mathbf{9}$ , we have

$$\vdash \mathbf{ax}_{\mathbf{k}} \iff [0] \Vdash \mathbf{ax}_{\mathbf{k}}$$

# Axioms for the interval

## Theorem ( $\mathbf{ax}_2$ )

$$[0] \Vdash \neg(0 = 1)$$

It suffices to show that for all  $[n]$ , we do not have  $[n] \Vdash 0 = 1$ . Assume  $[n] \Vdash 0 = 1$ , then we would have  $0 = 1 : \mathbf{y}[n] \rightarrow \mathbb{I}$ , which is false as  $0_n \neq 1_n$ .



# Axioms for the interval

## Theorem (ax<sub>2</sub>)

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## Theorem (ax<sub>3</sub>)

$$[0] \Vdash (\forall i : \mathbb{I}), 0 \sqcap i = 0 = i \sqcap 0 \wedge 1 \sqcap i = i = i \sqcap 1$$

Call  $\sigma(i) = 0 \sqcap i = 0 = i \sqcap 0 \wedge 1 \sqcap i = i = i \sqcap 1$ . We want to show that  $[n] \Vdash \sigma(i)$  for all  $f : [n] \rightarrow [0]$  and  $i : \mathbf{y}[n] \rightarrow \mathbb{I}$ . Such a map  $f$  is unique, thus we need to show that  $[n] \Vdash \sigma(i)$ , for all  $i : \mathbf{y}[n] \rightarrow \mathbb{I}$ . Unwinding of the  $\wedge$  connective, it suffices to prove each equality independently. For instance with  $[n] \Vdash 0 \sqcap i = 0$ , let  $i : \mathbf{y}[n] \rightarrow \mathbb{I}$  be a generalized element. It suffices to show that  $0 \sqcap i = 0$  as maps of  $\mathcal{E}$ , and this is true.

# The interval is connected

## Theorem

Let  $\varphi, \psi : \mathbb{I} \rightarrow \Omega$  be two formulas. Then the following are equivalent.

- (i)  $\mathbb{I} \vdash \psi \vee \varphi$
- (ii)  $\mathbb{I} \vdash \psi$  or  $\mathbb{I} \vdash \varphi$

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## Theorem (**ax<sub>1</sub>**)

$$\vdash \forall \varphi : \mathbb{I} \rightarrow \Omega, (\forall i : \mathbb{I}, \varphi i \vee \neg \varphi i) \Rightarrow (\forall i : \mathbb{I}, \varphi i) \vee (\forall i : \mathbb{I}, \neg \varphi i)$$

# Axioms for cofibrations

For the cofibration, we prove alternative characterization of the axioms.

## Theorem (Characterization for cofibrations)

*If the class  $\mathbf{Cof} \in \mathcal{E}$  of cofibrant propositions is such that*

- $\{i : \mathbb{I} \mid i = 0\} \twoheadrightarrow \mathbb{I}$  *is a cofibration*
- $\{i : \mathbb{I} \mid i = 1\} \twoheadrightarrow \mathbb{I}$  *is a cofibration*
- *If  $f : X \twoheadrightarrow Y$  and  $g : Y \twoheadrightarrow Z$  are cofibrations, then  $g \circ f$  is a cofibration*
- *If  $f : X \twoheadrightarrow Y$  is a cofibration, then  $f^{\mathbb{I}} : X^{\mathbb{I}} \twoheadrightarrow Y^{\mathbb{I}}$  is a cofibration*

*then  $\vdash \mathbf{ax}_k$ , for  $k = 5, 7, 8$*

# The axiom of strictification

## Definition (Decidable sieves)

We define  $\Omega_{\text{dec}} \rightarrow \Omega$  to be the sub-presheaf consisting of decidable sieves, that is

$$\Omega_{\text{dec}}(c) = \{S \in \Omega(c) \mid S \text{ is a decidable subset of } \text{Obj}(\mathcal{C}/c)\}$$

# The axiom of strictification

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We define  $\Omega_{\text{dec}} \multimap \Omega$  to be the sub-presheaf consisting of decidable sieves, that is

$$\Omega_{\text{dec}}(c) = \{S \in \Omega(c) \mid S \text{ is a decidable subset of } \text{Obj}(\mathcal{C}/c)\}$$

## Theorem

**ax<sub>9</sub>** *is satisfied in  $\mathcal{E}$  if  $\text{Cof} \multimap \Omega$  is contained in  $\Omega_{\text{dec}} \multimap \Omega$ .*

# Main theorem

## Theorem (Model of HoTT)

*If  $\mathcal{E} = [\square^{\text{op}}, \text{Sets}]$  with  $\mathbb{I} = \mathbf{y}[1]$  and  $\text{Cof} = \Omega_{\text{dec}}$ , then  $\vdash \mathbf{ax}_k$  for  $k = \mathbf{1}, \dots, \mathbf{9}$ , thus its internal type theory is a model of cubical type theory with univalence.*

# Main theorem

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We did not use the same cofibrations as in [CCHM15]. They use the face lattice  $\mathbb{F}$ , seen as an object of  $\mathcal{E}$ , and define the mono  $m : \mathbb{F} \rightarrow \Omega$  by sending  $x \in \mathbb{F}([n])$  to the sieve

$$m_{[n]}(x) = \{ \cdot \xrightarrow{f} [n] \mid \mathbb{F}(f)(x) = 1 \}$$

and they take  $\text{Cof} \rightarrow \Omega$  to be the subobject given by this monomorphism  $m$ .



# Conclusion





# Summary

- Lack of computational content in e.g. simplicial models.
- In cubical settings, we can compute the univalence axiom, but the syntax of the cubical type theory tends to be technical.
- From the semantic point of view, there is not one *good* category of cubes [Mö21].


# Summary

- Lack of computational content in e.g. simplicial models.
- In cubical settings, we can compute the univalence axiom, but the syntax of the cubical type theory tends to be technical.
- From the semantic point of view, there is not one *good* category of cubes [Mö21].
- We focused on the cubes of [CCHM15] and proved that it is indeed a model of this specific cubical type theory.
- We did that thanks to [OP16], with the nine axioms.
- But we needed to answer the question of what it means for an axiom to be true in a topos.
- Thus Kripke-Joyal forcing.
- And Kripke-Joyal forcing in the context of categories with families, as in [AGH21].
- This provides a more systematic approach that could be generalized to various presheaf toposes.

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Thanks!