Cubical Type Theory Inside a Presheaf Topos

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SYCO10

Master's thesis
Supervised by Pierre-Louis Curien

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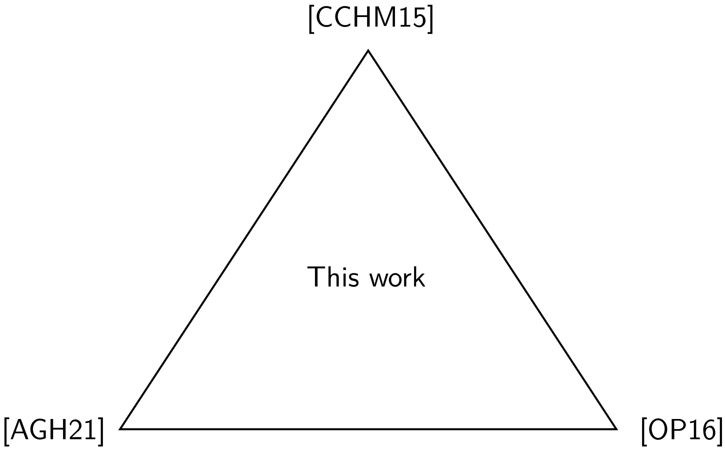
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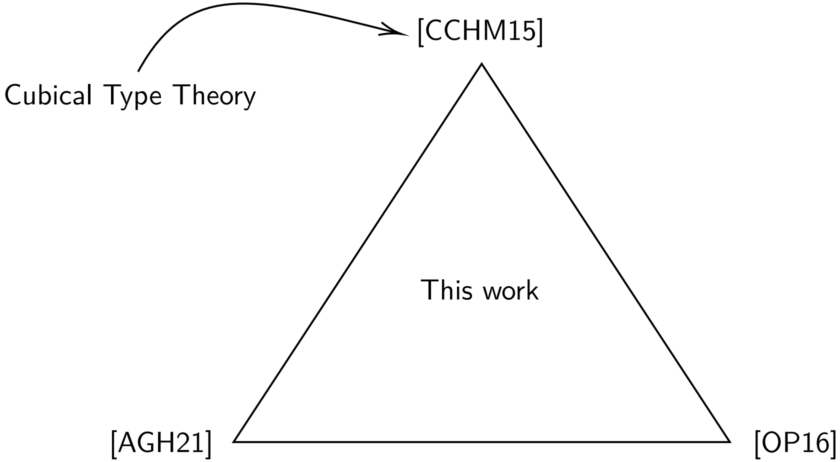
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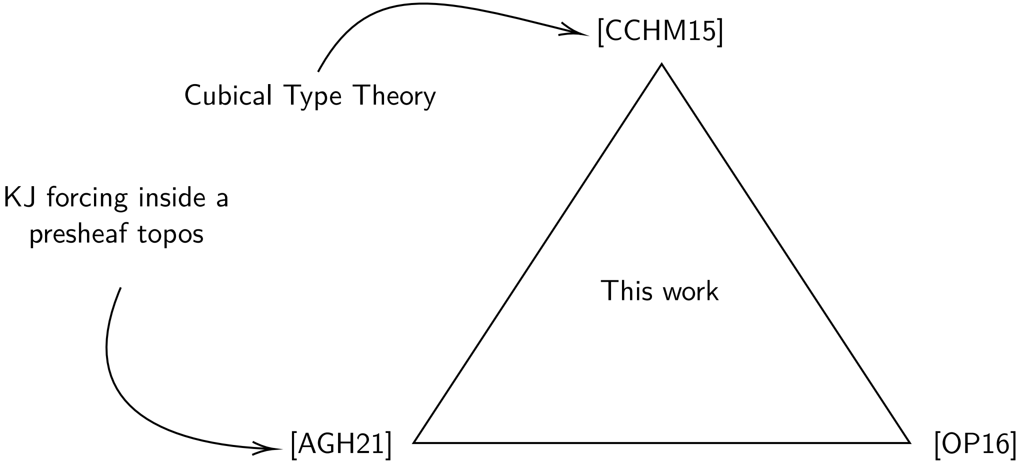
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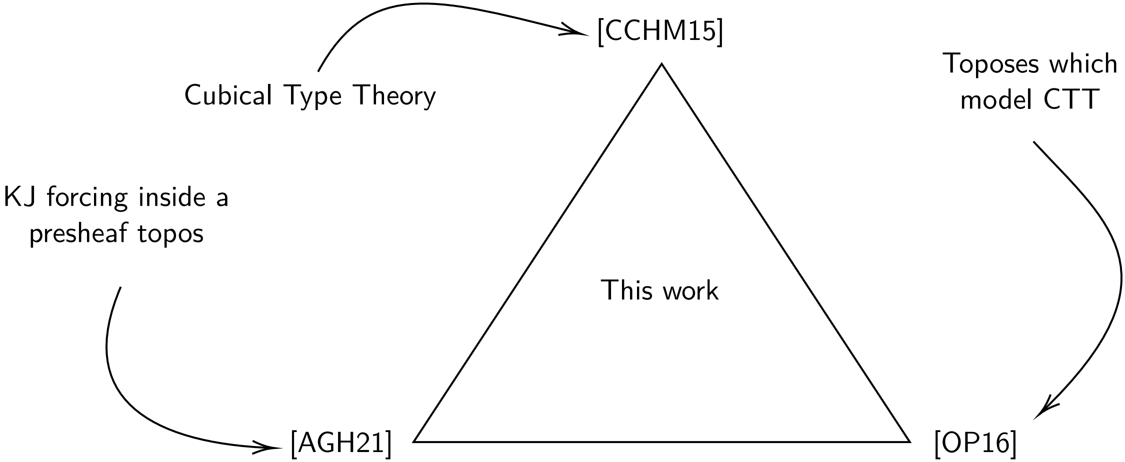

[CCHM15]
Cubical Type Theory
Toposes which model CTT
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Cubical Type Theory

Rules and interval

1. Standard Martin-Löf dependent type theory
2. An object $\mathbb{I}$ which is the free de Morgan algebra on a fixed infinite set of names $i, j, k, \ldots$
3. Grammar of $\mathbb{I}$ is

$$r, s ::= 0 \mid 1 \mid i \mid \neg r \mid r \wedge s \mid r \vee s$$

4. Custom λ-abstraction for $i : \mathbb{I}$:

$$\langle i \rangle . t$$

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$$\begin{array}{llll}
\Gamma, \Delta & ::= & () \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} & \text{Contexts} \\
t, u, A, B & ::= & x \mid \lambda x : A.t \mid t\ u \mid (x : A) \to B & \Pi\text{-types} \\
 & \mid & (t, u) \mid t.1 \mid t.2 \mid (x : A) \times B & \Sigma\text{-types} \\
 & \mid & \mathtt{Path}\ A\ t\ u \mid \langle i \rangle\ t \mid t\ r & \text{Path types}
\end{array}$$

Interval

Insight on $\mathbb{I}$

1. $\mathbb{I}$ is a synthetic equivalent for $[0, 1]$.
2. $\vee$ represents max
3. $\wedge$ represents min
4. $\neg$ represents $1 - \cdot$
5. We write $(i0)$ and $(i1)$ for $(i/0)$ and $(i/1)$.

Jugdmental equalities for $\mathbb{I}$

$\neg 0 = 1$	$\neg 1 = 0$	$\neg(r \vee s) = \neg r \wedge \neg s$	$\neg(r \wedge s) = \neg r \vee \neg s$

Path types

Rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \texttt{Path}\ A\ a\ b}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle\ a : \texttt{Path}\ A\ a(i0)\ a(i1)}$$

$$\frac{\Gamma \vdash p : \texttt{Path}\ A\ a\ b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p\ r : A}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle\ a)\ r = a(i/r) : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash p\ i = q\ i : A}{\Gamma \vdash p = q : \texttt{Path}\ A\ p_0\ p_1}$$

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Path types

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$$\frac{\Gamma \vdash p : \mathtt{Path}\ A\ a\ b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p\ r : A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle\ a)\ r = a(i/r) : A}$$

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Consequences

1. Reflexivity: For $a : A$, $1_a = \langle i \rangle\ a : \mathtt{Path}\ A\ a\ a$
2. Function extensionality, from $\Gamma \vdash p : (x : A) \to \mathtt{Path}\ B\ (f\ x)\ (g\ x)$ we have

$$\Gamma \vdash \langle i \rangle\ \lambda x : A.\ p\ x\ i : \mathtt{Path}\ ((x : A) \to B)\ f\ g$$

Geometric intuition

In dimension n

n variables of dimension $i_1, \ldots, i_n : \mathbb{I}$ in the context, correspond to an n-dimensional cube.

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In dimension two

$$i : \mathbb{I}, j : \mathbb{I} \vdash A$$

$$\begin{array}{ccc}
A(i0)(j1) & \xrightarrow{A(j1)} & A(i1)(j1) \\
{\scriptstyle A(i0)}\uparrow & & \uparrow{\scriptstyle A(i1)} \\
A(i0)(j0) & \xrightarrow[A(j0)]{} & A(i1)(j0)
\end{array}$$

Face lattice

Definition (Face lattice)

We define $\mathbb{F}$ to be the distributive lattice generated by the symbols $(i = 0)$ and $(i = 1)$ (for all dimension name i) with relation $(i = 0) \wedge (i = 1) = 0_{\mathbb{F}}$. The grammar is

$$\phi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \phi \wedge \psi \mid \phi \vee \psi$$

We have the rule

$$\frac{\Gamma \vdash \phi : \mathbb{F}}{\Gamma, \phi \vdash}$$

Contractible types and equivalences

Definition (Contractible types)

A type A is contractible if

$$\texttt{isContr}\ A \triangleq (x : A) \times ((y : A) \to \texttt{Path}\ A\ x\ y)$$

is inhabited.

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is inhabited.

Definition (Equivalence)

Given two types T, A and $f : T \to A$, we define

$$\texttt{isEquiv}\ T\ A\ f \triangleq (y : A) \to \texttt{isContr}\ ((x : T) \times \texttt{Path}\ A\ y\ (f\ x))$$

We define the type

$$\texttt{Equiv}\ T\ A \triangleq (f : T \to A) \times \texttt{isEquiv}\ T\ A\ f$$

Glueing

Definition (Glueing)

The glueing operation allows us to transport types along an equivalence. The formation rule is:

$$\frac{\Gamma \vdash A \qquad \Gamma, \phi \vdash T \qquad \Gamma, \phi \vdash f : \texttt{Equiv}\ T\ A}{\Gamma \vdash \texttt{Glue}\ [\phi \mapsto (T, f)]\ A}$$

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Intuition for glueing

If

1. $\Gamma \vdash A$
2. $\Gamma, \phi \vdash T$
3. A and T are equivalent on the region ϕ

then we have the equality $\Gamma, \phi \vdash \texttt{Glue}\ [\phi \mapsto (T, f)]\ A = T$.

Glueing and univalence

Theorem (Univalence in Cubical Type Theory)

For any term

$$t : (A\ B : U) \to \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$$

the map $t\ A\ B : \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$ *is an equivalence.*

Logic and type theory of a (presheaf) topos

Kripke-Joyal semantics

Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.

Kripke-Joyal semantics

Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.
- Then, we use the notion of Kripke-Joyal forcing to recursively unwind formulas,
- Thus transforming a formula into a a lot of little pieces.

A presheaf category is a topos

We now work in $\mathcal{E} = [\mathcal{C}^{\mathsf{op}}, \mathrm{Sets}]$, a presheaf category. It has a topos structure by letting Ω_c to be set of sub-functors of $\mathbf{y}c$.

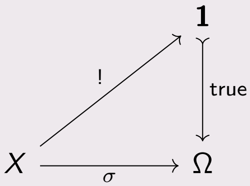
We interpret $\sigma : X \to \Omega$ as a formula in context X, and the following is a pullback:

$$\begin{array}{ccc} \{x : X \mid \sigma\} & \longrightarrow & \mathbf{1} \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{true}} \\ X & \xrightarrow[\sigma]{} & \Omega \end{array}$$

Validity of a formula

Definition (Validity)

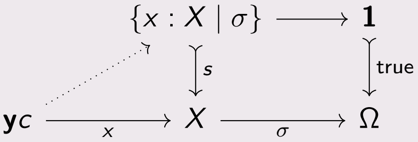
Let $\sigma : X \to \Omega$. We say that σ is *valid* whenever σ factors through $\text{true} : \mathbf{1} \to \Omega$.



In that case, we write $X \vdash \sigma$. If σ is a closed formula, then we write $\vdash \sigma$ and this amounts to say that $\sigma = \text{true}$.

Definition (Forcing)

Let $\sigma : X \to \Omega$ be a formula and $x : \mathbf{y}c \to X$. We say that c *forces* $\sigma(x)$, written $c \Vdash \sigma(x)$, if the following dotted arrow exists, making the left triangle commute.



Kripke-Joyal forcing

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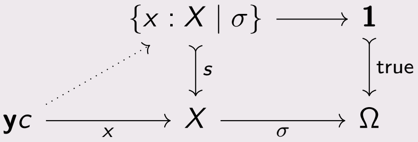

Theorem

Let $\sigma : X \to \Omega$. $X \vdash \sigma$ if and only if $c \Vdash \sigma(x)$ for all $x : \mathbf{y}c \to X$.

The main theorem

Theorem (Conditions for forcing)

Let $\sigma, \tau : X \to \Omega$, $\theta : Y \times X \to \Omega$ *and* $x : \mathbf{y}c \to X$, *then*

- $c \Vdash \bot$ *never*
- $c \Vdash \top$ *always*
- $c \Vdash \sigma(x) \wedge \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *and* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \vee \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *or* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \Rightarrow \tau(x)$ *if and only if for all* $f : d \to c$, $d \Vdash \sigma(xf)$ *implies* $d \Vdash \tau(xf)$
- $c \Vdash \neg\sigma(x)$ *if and only if for all* $f : d \to c$, *we do not have* $d \Vdash \sigma(xf)$
- $c \Vdash \exists y : Y,\ \theta(y, x)$ *if and only if* $c \Vdash \theta(y, x)$ *for some* $y : \mathbf{y}c \to Y$
- $c \Vdash \forall y : Y,\ \theta(y, x)$ *if and only if* $d \Vdash \theta(y, xf)$ *for all* $f : d \to c$ *and* $y : \mathbf{y}d \to Y$

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We can recursively unwind the connectives in $c \Vdash \sigma(x)$

Update to Type Theory

Overview

General setting

We will follow [AGH21] to define small maps, and the small map classifier π given by a Hofmann-Streicher lifting of a Grothendieck universe.

- We fix κ a (strongly) inaccessible cardinal.
- A set *small* if it has cardinality less than κ.
- We write Sets_κ for the full subcategory of Sets consisting of small sets (which is a Grothendieck universe).
- We fix a small (in the above sense) category $\mathcal{C}$.
- We call $\mathcal{E}$ the associated presheaf topos.

Hofmann-Streicher lifting

Definition (Small maps)

1. We say that a presheaf $A \in \mathcal{E}$ is *small* if $A(c)$ is a small set, for all $c \in \mathcal{C}$
2. We say that $p : A \to X$ in $\mathcal{E}$ is a *small map* if, for every $x : \mathbf{y}c \to X$, the presheaf A_x obtained by the pullback

$$\begin{array}{ccc} A_x & \longrightarrow & A \\ \downarrow & & \downarrow p \\ \mathbf{y}c & \xrightarrow[x]{} & X \end{array}$$

is small.

Hofmann-Streicher lifting

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is small.

Class of small maps

We call $\mathcal{S}$ the class of small maps in $\mathcal{E}$. In the same way that we can classify the monos $S \rightarrowtail X$ with the map $\mathrm{true} : \mathbf{1} \to \Omega$, we define $\pi : E \to U$ that classifies the maps of $\mathcal{S}$.

Classification

Given a small map $p : A \to X$, there exists a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[c_p]{} & U \end{array}$$

We say that p is *classified* by c_p. Conversely, we introduce a canonical pullback p_A for each $A : X \to U$:

$$\begin{array}{ccc} X.A & \longrightarrow & E \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[A]{} & U \end{array}$$

The map p_A is called the *display map* of A.

Category with families

Lemma (Category with families)

The presheaf category $\mathcal{E}$ determines a category with families, as follows:

- *The contexts are the objects $X \in \mathcal{E}$*
- *A type A in context X is a map $A : X \to U$*
- *A term $a : A$ in context X is a map $a : X \to E$ such that the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{a} & E \\ \| & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[A]{} & U \end{array}$$

- *Definitional equality on terms or types is defined via equality of maps in the topos. For instance, we have $X \vdash A = B$ if and only if $A : X \to U$ and $B : X \to U$ are the same maps in $\mathcal{E}$. For terms, we write $X \vdash a = b : A$.*

Extending the forcing

Definition

Let $A : X \to U$ be a type in context X, and $x : \mathbf{y}c \to X$. For $a : \mathbf{y}c \to E$, We say c *forces* $a : A(x)$ written $c \Vdash a : A(x)$ if the following diagram commutes.

$$
\begin{array}{ccc}
\mathbf{y}c & \xrightarrow{\;a\;} & E \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \pi} \\
X & \xrightarrow[\;A\;]{} & U
\end{array}
$$

Like in the standard forcing, $c \Vdash a : A(x)$ is to say $\mathbf{y}c \vdash a : A(x)$:

$$
\begin{array}{ccccc}
\mathbf{y}c & & \xrightarrow{\qquad a \qquad} & & E \\
\Big\| & & & & \downarrow{\scriptstyle \pi} \\
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Theorem for the forcing

Alternative point of view

Let $A : X \to U$ be a type in context X, and $x : \mathbf{y}c \to X$. An element $a : \mathbf{y}c \to E$ is the same thing as the dotted arrow in the following diagram:

$$\begin{array}{ccccc} & & X.A & \xrightarrow{q_A} & E \\ & \overset{u}{\nearrow} & \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle \pi} \\ \mathbf{y}c & \xrightarrow[x]{} & X & \xrightarrow[A]{} & U \end{array}$$

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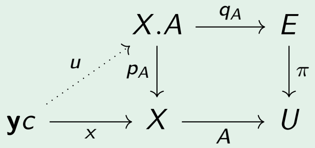

Theorem

The data of $a : X \to E$ such that $X \vdash a : A$ is the same as families of elements $a_x : \mathbf{y}c \to E$ such that $c \Vdash a_x : A(x)$, and are uniform in the sense that $c \Vdash a_x = a(x) : A(x)$.

Main theorem

Theorem (Conditions for forcing)

1. $c \Vdash a : 0$ *never*
2. $c \Vdash a : 1$ *for a unique* $a = \star : \mathbf{y}c \to E$
3. $c \Vdash t : (A + B)(x)$ *if and only if* $c \Vdash a : A(x)$ *with* $t = \mathrm{inl}(a)$ *or* $c \Vdash b : B(x)$ *with* $t = \mathrm{inr}(a)$
4. $c \Vdash (a, b) : \Sigma_A B(x)$ *if and only if* $c \Vdash a : A$ *and* $c \Vdash b : B(a)$
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Theorem (Relation to the old forcing)

Let $\sigma : X \to \Omega$ *be a proposition and* $x : \mathbf{y}c \to X$*. Then the following are equivalent.*

(i) $c \Vdash_{old} \sigma(x)$

(ii) $c \Vdash s : \{\sigma(x)\}$ *for a (necessarily unique)* $s : \mathbf{y}c \to E$

Cubical Type Theory of a topos

Cubical presheaves

- We introduce the category $\square$ (from [CCHM15]). It will be the base category of a presheaf topos whose internal type theory will model cubical type theory.
- Then, we introduce the notion of cofibration, whose behavior is important to internalize glueing [OP16].

The box category

For $n \geq 0$, we denote by I_n the free de Morgan algebra on n generators.

Definition ($\square$)

We call $\square$ the category having as objects cardinal numbers $[n] \geq 0$ and as morphisms in $\square([n],[m])$ the de Morgan homomorphisms $f : I_m \to I_n$.

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We call $\Box$ the category having as objects cardinal numbers $[n] \geq 0$ and as morphisms in $\Box([n],[m])$ the de Morgan homomorphisms $f : I_m \to I_n$.

The interval

We take $\mathbb{I} \triangleq \mathbf{y}[1]$, it has a de Morgan structure defined pointwise.

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The interval

We take $\mathbb{I} \triangleq \mathbf{y}[1]$, it has a de Morgan structure defined pointwise.

Theorem

$\square$ has finite products.

Theorem

For all $[n] \in \square$, $\mathbb{I}_n$ has decidable equality.

Cofibrations

Idea

Cofibrations are useful for glueing, and are the way to semantically specify regions of the n-dimensional cube. We assume a map $\mathsf{cof} : \Omega \to \Omega$ and we consider the associated subobject

$$\texttt{Cof} = \{\phi : \Omega \mid \mathsf{cof}\,\phi\}$$

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A *cofibration* is a monomorphism whose classifying arrow factors trough $\mathtt{Cof} \rightarrowtail \Omega$.

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Theorem

Let $\phi : X \to \Omega$ be a proposition. For every $x : \mathbf{y}c \to X$, the following are equivalent.

1. $c \Vdash \mathsf{cof}\,\phi(x)$
2. $\phi \circ x : \mathbf{y}c \to \Omega$ *is a cofibration*

Axioms to model CTT

Theorem ([OP16])

The category with families of a presheaf topos is a model of cubical type theory if it has two objects $\mathbb{I}$ and $\texttt{Cof}$ such that its internal logic satisfies the nine following axioms.

ax$_1$ $\forall \phi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \phi\ i \vee \neg \phi\ i) \Rightarrow (\forall i : \mathbb{I},\ \phi\ i) \vee (\forall i : \mathbb{I},\ \neg \phi\ i)$

ax$_2$ $\neg(0 = 1)$

ax$_3$ $\forall i : \mathbb{I},\ 0 \sqcap i = 0 = i \sqcap 0\ \wedge\ 1 \sqcap i = i = i \sqcap 1$

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Kripke-Joyal forcing to prove the axioms

To prove each of these axioms, we can use Kripke-Joyal forcing and unravel each formula. Moreover,

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Theorem (Forcing with a terminal object)

If $\mathcal{C}$ has a terminal object $t \in \mathcal{C}$, then a closed formula $\sigma : \mathbf{1} \to \Omega$ is valid if and only if $t \Vdash \sigma$.

Forcing in $\square$

Since $\square$ has a terminal object $[0]$, it suffices to prove that each axiom is forced at stage $[0]$. That is, for $\mathbf{k} = \mathbf{1}, \ldots, \mathbf{9}$, we have

$$\vdash \mathbf{ax_k} \iff [0] \Vdash \mathbf{ax_k}$$

The interval is connected

Lemma

Let $\phi, \psi : \mathbb{I} \to \Omega$ be two formulas. Then the following are equivalent.

1. $\mathbb{I} \vdash \psi \vee \phi$
2. $\mathbb{I} \vdash \psi$ *or* $\mathbb{I} \vdash \phi$

Proof.

Recall that $\mathbb{I} = \mathbf{y}[1]$.

$$\mathbf{y}[1] \vdash \psi \vee \phi \iff [1] \Vdash \psi \vee \phi \iff [1] \Vdash \psi \text{ or } [1] \Vdash \phi \iff \mathbf{y}[1] \vdash \psi \text{ or } \mathbf{y}[1] \vdash \phi$$

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Thus, $\mathbf{ax_1} : \vdash \forall \phi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \phi\ i \vee \neg \phi\ i) \Rightarrow (\forall i : \mathbb{I},\ \phi\ i) \vee (\forall i : \mathbb{I},\ \neg \phi\ i)$.

Axioms for the interval

Theorem ($\mathbf{ax_2}$)

$[0] \Vdash \neg(0 = 1)$

It suffices to show that for all $[n]$, we do not have $[n] \Vdash 0 = 1$. Assume $[n] \Vdash 0 = 1$, then we would have $0 = 1 : \mathbf{y}[n] \to \mathbb{I}$, which is false as $0_n \neq 1_n$.

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Theorem ((Part of) $\mathbf{ax_3}$)

$[0] \Vdash (\forall i : \mathbb{I}),\ 0 \sqcap i = 0 \wedge i \sqcap 0 = 0$

By $\forall$-forcing, it is equivalent to show that $[n] \Vdash 0 \sqcap i = 0 \wedge i \sqcap 0 = 0$ for all $f : [n] \to [0]$ and $i : \mathbf{y}[n] \to \mathbb{I}$. Such a map f is unique, thus we need to show that $[n] \Vdash 0 \sqcap i = 0 \wedge i \sqcap 0 = 0$, for all $i : \mathbf{y}[n] \to \mathbb{I}$.

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Main theorem

Theorem (Model of HoTT)

If $\mathcal{E} = [\square^{\mathsf{op}}, \mathrm{Sets}]$ *with* $\mathbb{I} = \mathbf{y}[1]$ *and* $\mathtt{Cof} = \Omega_{\mathtt{dec}}$, *then* $\vdash \mathbf{ax_k}$ *for* $\mathbf{k} = \mathbf{1}, \ldots, \mathbf{9}$, *thus its internal type theory is a model of cubical type theory with univalence.*

Summary and future work

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Context

- Lack of computational content in e.g. simplicial models.
- In cubical settings, we can compute the univalence axiom, but the syntax of the cubical type theory tends to be technical, and from the semantic point of view, there is not one *good* category of cubes [Mö21].

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This work

- KJ forcing to prove that $[\square^{\mathsf{op}}, \mathrm{Sets}] \vDash \mathbf{ax_k}$.
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Future directions

- Improve the forcing theorem with more of CTT (W-types, higher inductive types, etc.).
- Generalize to a larger class of toposes.

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Thanks!