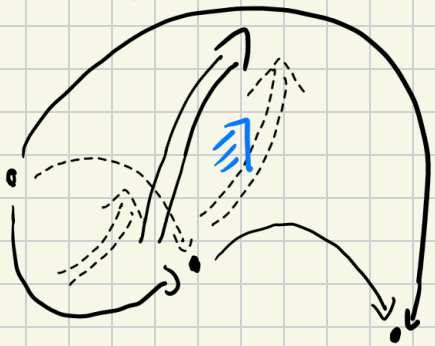
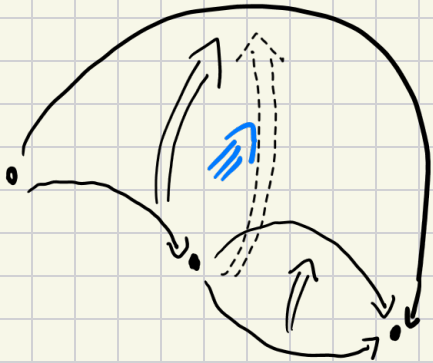
We consider the following four 3-dimensional atoms A^+, B^+, B^-, and A^-.

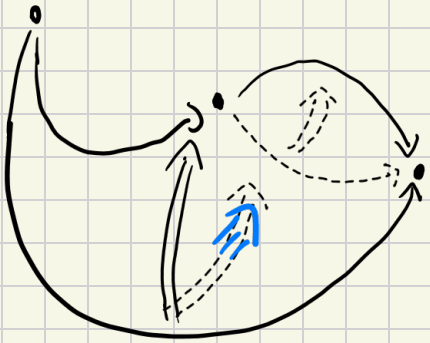
A^+:



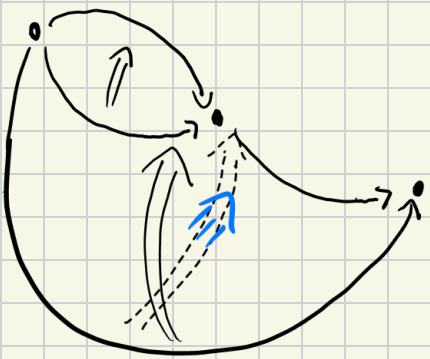
: B^-



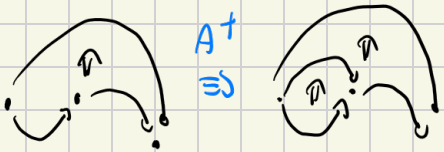
B^+:

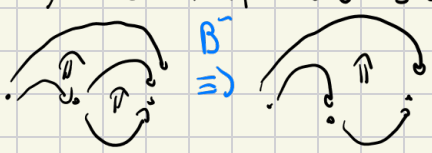


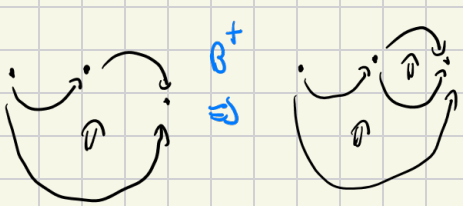
: A^-

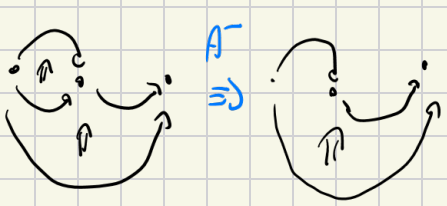


Since 3d drawings are difficult to see, we may draw instead:



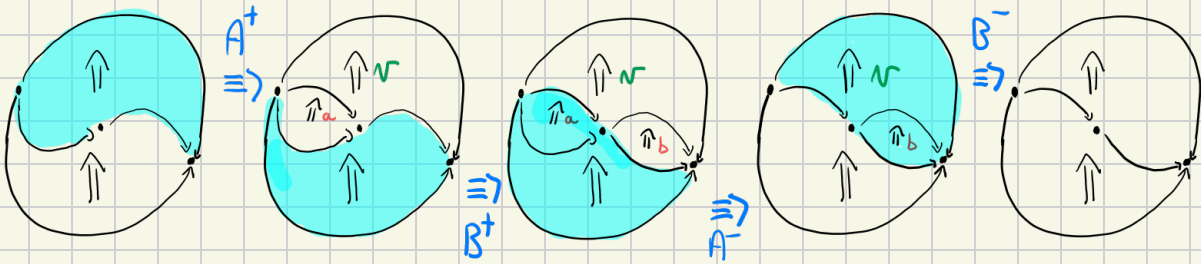






Then, we may consider the molecule U given by $U := (B^+ \#_1 A^+) \#_2 (B^- \#_1 A^-)$.

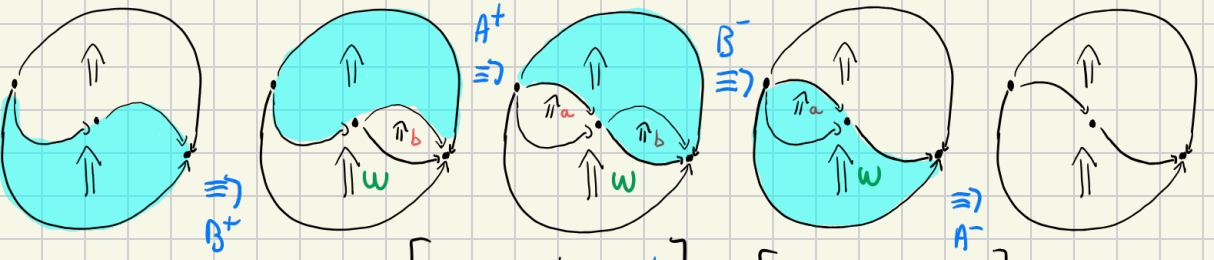
This molecule U can also be written as:



Let us call $S := [B^+ \#_1 (a \#_0 \delta_1^+ b)] \#_2 [(\delta_1^+ a \#_0 b) \#_1 A^-]$.

Then, $U = [\delta_2 B^+ \#_1 A^+] \#_2 [S \#_1 N] \#_2 [\delta_2^+ A^- \#_1 B^-]$.

The molecule U can also be written as:



Let us call $T := [(\delta^- a \#_0 b) \#_1 A^+] \#_2 [(a \#_0 \delta_1^- b) \#_1 B^-]$.

Then $U = [B^+ \#_1 \delta_2^- A^+] \#_2 [w \#_1 T] \#_2 [A^- \#_1 \delta_2^+ B^-]$.

Now, S and T are both submolecules of U, but not "at the same time" (even though S and T have no common 3-cell). If one consider the first decomposition of U, there is no way to make T appear while fixing S:

$$U = \left[\partial_2^- B^+ \#_1 A^+\right] \#_2 \left[S \#_1 \nu\right] \#_2 \left[\partial_2^+ A^- \#_1 B^-\right]$$

half of T is here — and — the other half is there, but neither halves can move around S. Indeed, A^+ cannot go after S since A^- is in S, and A^- needs A^+ to be done before existing. Similary, B^- cannot go before S because of B^+ in S.

A dual argument apply for the second decomposition

$$U = \left[B^+ \#_1 \partial_2^- A^+\right] \#_2 \left[w \#_1 T\right] \#_2 \left[A^- \#_1 \partial_2^+ B^-\right].$$

the two halves of S cannot be joined.

Next, the molecules S and T are round, so we may consider the 4-dimensional atoms:

$\sigma := S \Rightarrow S$, and $\tau := T \Rightarrow T$. Then,

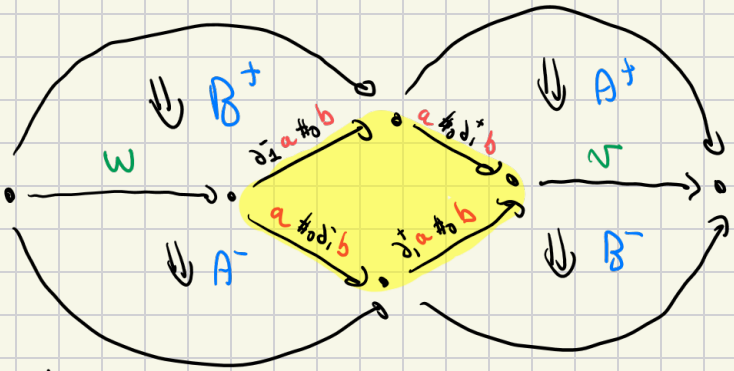
we formally replace S and T by σ and τ in two copies of U, that is, we define

$$U_\sigma := \left[\partial_2^- B^+ \#_1 A^+\right] \#_2 \left[\sigma \#_1 \nu\right] \#_2 \left[\partial_2^+ A^- \#_1 B^-\right], \text{ and}$$

$$U_\tau := \left[B^+ \#_1 \partial_2^- A^+\right] \#_2 \left[w \#_1 \tau\right] \#_2 \left[A^- \#_1 \partial_2^+ B^-\right].$$

Theorem: $U_\sigma \#_3 U_\tau = U_\tau \#_3 U_\sigma$, yet this equation is not provable using only exchange and associativity. This is a **higher-exchange**.

To unpack this, let Mol/U be the strict 3-category generated by U. Then the hom 2-category $\mathrm{Mol}/U(\partial_0^- U, \partial_0^+ U)$ looks like this:



The middle yellow part is where S and T intersect, which is a 2-dimensional "structural equality".

The molecule $U_\sigma \#_3 U_\tau$ first does σ which rewrites B^+ and A^- into copies of themselves. For that, it needs to "access" the yellow intersection, but crucially, it leaves it "unchanged". After σ is done, τ rewrites A^+ and B^- into copies of themselves and also need the yellow intersection that it leaves unchanged.

The higher-exchange states that we could have done first τ then σ and obtain the same result, i.e.

$$U_\sigma \#_3 U_\tau = U_\tau \#_3 U_\sigma .$$

It cannot be proven by pure exchange and associativity, since the yellow intersection cannot be accessed in parallel both by σ and τ.