# Notes on sheaf cohomology in Grothendieck toposes

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## Contents



Sheaf cohomology is a big topic. This is a more-or-less self contained note that produces sheaf cohomology on an arbitrary Grothendieck topos. We start by trying to explain the high-level way to define cohomology, and show informally how it is relevant to distinguish the circle from the line. Then we prove some general results on abelian categories, and finally, we build cohomology groups. The only main result that is not proven is Theorem 39, which is the reason why cohomology is an homotopical invariant.

All our functors are properly enriched.

# 1 General idea

Before entering in the detail, we give a rough outline of the construction of sheaf cohomology in a Grothendieck topos. Then we discuss what kind of behavior it can detect, and how it does that.

## 1.1 What is sheaf cohomology

As always with homological algebra, the goal is to find homotopical invariants of topological spaces. Sheaf cohomology allows us to be a little bit more general, and provides invariants for any Grothendieck site. A site $(\mathcal{C}, J)$ is a category $\mathcal{C}$ together with a topology $J$ that specifies *open covers* of objects of $\mathcal{C}$. More precisely, given an object $c \in \mathcal{C}$, we get a set $J_c$ whose elements are sieves on $c$. A sieve on $c$ is a family of morphisms with codomain $c$, closed under precomposition. Typically, if $(X, \mathcal{O})$ is a topological space, then seing $(\mathcal{O}, \subseteq)$ as a posetal category, a sieve on $U \in \mathcal{O}$ will be a collection of subsets of $U$, closed under inclusion. The axioms for Grothendieck topology will then ensure that such a sieve constitutes an open cover of $U$, and the canonical Grothendieck topology associated to any topological space is therefore the collection of all open covers. The recipe to do sheaf cohomology is as follow. We fix $\mathbb{A}$, an abelian category, i.e. a category where "abelian group theory" works, like, of course, the category of abelian groups, or that of commutative rings. First, we have the setup, it consists of two parts.

1. Select your base space, that is a Grothendieck site $(\mathcal{C}, J)$. Of course, now that we have a site, we get a topos for free, and we let $\mathbb{A}(\mathcal{C})$ be the associated Grothendieck topos.
2. Select an $\mathbb{A}$-valued sheaf $F : \mathcal{C}^{\text{op}} \to \mathbb{A}$, that is a presheaf, which is continuous with respect to the topology on $\mathcal{C}$. This sheaf acts as data that we put on our space. On each $c \in \mathcal{C}$, we have some data (an abelian object in $\mathbb{A}$) given by $Fc$, that behaves coherently with respect to the morphisms. This is the presheaf part of the sheaf. To be a sheaf, a presheaf needs to respect the sheaf condition. Roughly, it means that if a bunch of morphisms $\{c_i \to c\}_{i \in I}$ covers $c$ (think $U = \cup_i U_i$ in the case of topological spaces), then the data $\{Fc_i\}_i$ can be used to reconstruct exactly the data of $Fc$.

So we have an ambient space (a category $\mathcal{C}$) and some data associated to this space (a sheaf $F : \mathcal{C}^{\text{op}} \to \mathbb{A}$). This is all of the inputs of sheaf cohomology. We will soon produce objects of $\mathbb{A}$:

$$H^q(\mathcal{C}, F)$$

for each $q \in \mathbb{N}$, that will be our cohomology groups (or ring, etc, depending on $\mathbb{A}$).

Second, to define the cohomology groups, we proceed as follow. We first have to find an injective resolution of $F$, that is, an exact sequence

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots$$

where each $I^k$ is an injective sheaf, satisfying the lifting problem against all monomorphisms. Next, we apply the global section functor. The global section functor is $\mathcal{E}(1, -)$. It is called the global section for the following reason. Suppose that our sheaf topos $\mathbb{A}(\mathcal{C})$ comes from a topological space $(X, \mathcal{O})$, then the terminal sheaf is represented by $X$. Thus, applying the global section to a sheaf $F$ yields, via Yoneda, $F(X)$: the data that the sheaf associates to the whole space, thus the global data. Historically, the elements $s \in F(U)$ were called sections, as the sheaf one was working with, the structure sheaf, dealt with actual sections of some maps. The goal was then to find local sections of the maps, that is sections defined only on opens sets $U \in X$, and then look how this data can be patched together to form a global section on the whole $X$. This is exactly what sheaf cohomology measures.

So we have the global section functor $\mathcal{E}(1, -)$. For purely formal reason, it is left-exact. Indeed, it preserves, like all hom-functors, limits. However, it is not necessarily right-exact. And this defect of right-exactness is what we will measure. To do that, recall that we have an exact sequence

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots$$

Applying $\mathcal{E}(1, -)$ to it will give a sequence, that is exact if and only if the functor is right-exact. Thus in general, we will simply have a cochain complex

$$0 \to \mathcal{E}(1, F) \to \mathcal{E}(1, I^0) \to \mathcal{E}(1, I^1) \to \mathcal{E}(1, I^2) \to \dots$$

We do not really care about $0 \to \mathcal{E}(1, F)$, and simply want to measure the exactness of

$$0 \to \mathcal{E}(1, I^0) \to \mathcal{E}(1, I^1) \to \mathcal{E}(1, I^2) \to \dots$$

Now, as it is standard in homological algebra, we measure the defect of exactness by letting:

$$H^q(\mathcal{C}, F) := \frac{\ker(\mathcal{E}(1, I^q) \to \mathcal{E}(1, I^{q+1}))}{\mathrm{im}(\mathcal{E}(1, I^{q-1}) \to \mathcal{E}(1, I^q))}.$$

If the sequence is exact at $q$, then the numerator and the denominator are isomorphic and the $q$th cohomology group is 0, else, there is some lack of exactness, whose structure is encoded in $H^q(\mathcal{C}, F)$.

## 1.2 Why a line is not a circle

We know how how to define cohomology groups, but this does not tell us what kind of topological information it gives on the space. Cohomology is a homotopical invariant, that is, two homotopic spaces will have the same cohomologies (the converse is highly false). This fact is hidden when we chose the injective resolutions, injective objects are fibrations in the appropriate model structure, thus we are simply doing a fibrant replacement of our sheaf of coefficients. So, why a line is not a circle?

The presentation above was quite abstract and general. In fact, we can specialize this a little bit, and get a feel of more hands-on problems. Suppose we have $G \to H$, an epimorphism of sheaves. It is an interesting notion. For a map of sheaves to be an epi, it is not necessary that $Gc \to Hc$ is epi for all $c$, it merely suffices that is is epi *locally*, that is for all $y \in Hc$, there is a cover $\{c_i \to c\}_i$ of $c$ such that the restriction of $y$ to $c_i$ is in the image of $Gc_i \to Fc_i$. Then, it may well be the case that there is some section $s \in Hc$ that is the image of no $t \in Gc$ trough the map $Hc \to Gc$, even though, the restriction of the section $s$ to each $c_i$ is the image of some $t_i$ in $Gc_i$. This is why a line is not a circle, and this is something that cohomology detects. Let us be more precise, and compute the first cohomology group of the line and of the circle, with constant coefficients, and see where the above phenomenon happens for the circle and gives a non trivial first cohomology group.

We need some theory first. Suppose we have the following exact sequence of sheaves:

$$0 \to F \to G \to H \to 0.$$

This means exactly that the map $F \to G$ is mono while $G \to H$ is epi. General results of homological algebra yields that, in this case, there is an exact sequence of cohomology

$$\dots \to H^q(\mathcal{C}, F) \to H^q(\mathcal{C}, G) \to H^q(\mathcal{C}, H) \to H^{q+1}(\mathcal{C}, F) \to \dots$$

This is great. If we can control the cohomology of $G$ and $H$, then exactness will give us the cohomology of $F$. A very common thing to do is to choose the sheaf $G$ to be flabby. A sheaf is flabby whenever we can always reconstruct global sections from local sections, thus such a sheaf has vanishing cohomology in positive degree. It is also quite easy to compute zero-th cohomology groups, indeed by definition we have:

$$H^0(\mathcal{C}, F) = \mathcal{E}(1, \mathcal{C})$$

and if $\mathcal{C}$ has a terminal object 1, then this is $F(1)$ by Yoneda. So suppose we want to compute some first cohomology group

$$H^1(\mathcal{C}, F).$$

What we just describe tells us that if we can find $F \hookrightarrow G$ with $G$ flabby, then we are done. Indeed, take

$$0 \to F \hookrightarrow G \twoheadrightarrow G/F \to 0$$

where $G/F$ is the quotient sheaf. This is an exact sequence. Then by the above remark we have a chunk of exact sequence

$$H^0(\mathcal{C}, G) \to H^0(\mathcal{C}, G/F) \to H^1(\mathcal{C}, F) \to H^1(\mathcal{C}, G).$$

and we know that $G$ is flabby, thus $H^1(\mathcal{C}, G) = 0$, and we know what are the zero-th cohomology groups, hence our exact sequence becomes

$$G(1) \to G/F(1) \to H^1(\mathcal{C}, F) \to 0,$$

which is to say that

$$H^1(\mathcal{C}, F) \simeq \text{coker}(G(1) \to G/F(1)).$$

Let us do that with the line and the circle. Let us call $X$ to be either the line $[0,1]$, or the circle $S^1$. We wish to compute

$$H^1(X, \mathbf{Z}),$$

where $\mathbf{Z}$ is the constant sheaf. As our space is connected, the constant sheaf really act as the constant presheaf. The category $\mathcal{C}$ is the posetal category where objects are open sets $U \subseteq X$, and morphisms are inclusions. We construct the following presheaf

$$G(U) := \{f : U \to \mathbb{R} \mid f \text{ continuous}\},$$

and morphisms are given by restrictions. It is in fact a sheaf, for any open cover $U = \bigcup_i U_i$, if we have functions $f_i : U_i \to \mathbb{R}$ such that $f_i$ and $f_j$ agree on $U_i \cap U_j$, then we can patch this data into one unique function $f : \bigcup_i U_i \to \mathbb{R}$. We embed

$$F \hookrightarrow G$$

by seeing $n \in \mathbf{Z}$ as the constant function $n \in \mathbb{R}$. What is left to do is compute the quotient sheaf $G/F$, this is where everything happens. It is at this step that the structure of the open sets will create some interesting behaviors. Indeed, colimits of sheaves are *not* computed pointwise. We first need to forget that we are dealing with sheaves, and compute the colimits pointwise with our sheaves seen as presheaves. We then obtain a presheaf, and it is the sheafification of it that will be the colimit sheaf.

So computing the quotient $G/F$ seen as presheaf, we get simply

$$G/F(U) = \{f + \mathbf{Z} \mid f \in F(U)\}.$$

The sections of $G/F(U)$ are functions $f : U \to \mathbb{R}$ such that $f = f'$ if $f - f'$ is constant to an integer. Is this a sheaf? It depends on the structure of the open covers!

To understand that, we need to answer the following question. Let $f, g : U \to \mathbb{R}$ be functions. What does $f + \mathbf{Z} = g + \mathbf{Z}$ is telling us? Without loss of generality, let us rather compute what happens when $f + \mathbf{Z} = \mathbf{Z}$. In a first case, suppose $U$ is path connected, with center of contraction $u$. Then $f(u) = n \in \mathbf{Z}$. Now take any other point $v \in U$, take a path $\varphi : u \to v$, and assume that $f(v) \neq n$, meaning that we can consider $i = \inf\{j \mid f \circ \varphi(j) \neq n\}$, which is a strictly positive real number. By continuity, $\lim_{\varepsilon \to 0} f \circ \varphi(i \pm \varepsilon) = f \circ \varphi(i)$. If $f \circ \varphi(i) = n$, then we take the limit above with $i + \varepsilon$, otherwise we take the limit below with $i - \varepsilon$, and in both cases we reach a contraction. The conclusion of this little topological argument is that the sheaf condition propagates along paths. Now, in a second case, suppose we have a space $U = V \coprod W$, with, say, $V$ and $W$ path connected. Then by a similar argument, we prove $f + \mathbf{Z} = \mathbf{Z}$ is $f = n_V$ on $V$ and $f = n_W$ on $W$. Do we have $n_V = n_W$? No, because the two spaces $V$ and $W$ have no way to communicate, they are disjoint. A little bit more generally, we are saying that $f + \mathbf{Z} = g + \mathbf{Z}$, means that $f - g$ is constant on $U$, but this constant need to be the same only on each connected component. Let us call these choices of constants *degrees of freedom*. This is what happens during sheafification, let us see how this fact changes the cohomology.

We are now very close to the cohomology groups. The key point is that when we decompose $[0,1] = U \cup V$, with $U$ and $V$ two non-empty open intervals, then $U \cap V$ is contractible, thus the sheaf condition is that some $f_U + \mathbf{Z} = f_V + \mathbf{Z}$, and this will have one degree of freedom, on the contractible intersection. However, for the circle, if we write $S^1 = U \cup V$, with $U, V$ proper subsets, path connected, non empty, then $U \cap V$ is *always* made of two disconnected pieces. Thus the sheaf condition $f_U + \mathbf{Z} = f_V + \mathbf{Z}$ has two degrees of freedom. Then, loosely speaking, when we compute the cokernel, we realize that we can kill only one degree of freedom. Indeed, $G(U) \to G/F(U)$ sends a map $f$ to $f + \mathbf{Z}$, thus if $f + \mathbf{Z}$ is locally constant, then we can choose $f$ to be only one of these constants, on only one of the connected piece. Thus, (and we can show that with precise computations) one proves that $H^1(X, \mathbf{Z}) = \mathbf{Z}^{k-1}$ where $k$ is the degree of freedom, and the minus one stands for the one that we can kill by the surjection. That is,

$$H^1([0,1], \mathbf{Z}) = 0$$
$$H^1(S^1, \mathbf{Z}) = \mathbf{Z}.$$

# 2 Abelian categories

With these ideas in mind, we build some general results of abelian category, that will be useful to define general cohomologies in the next section.

## 2.1 First definitions

**Definition 1:** An *additive category* is an **Ab** enriched category with finite coproducts. More precisely, it is a category $\mathcal{C}$ such that for all $c, d \in \mathcal{C}$, $\mathcal{C}(a, b)$ has a structure of abelian group, and the compositions:

$$\mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)$$

are bilinear.

**Remark 2:** For all $c, d$ in an additive category, we always have the zero morphism $0_{c,d} \in \mathcal{C}(c, d)$, which is the identity of the abelian group.

**Lemma 3:** In a additive category, for all $f : c \to d$, we have $0_d \circ f = 0_{c,d}$ and $f \circ 0_c = 0_{c,d}$.

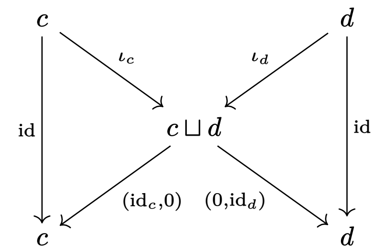
*Proof.* (**Lemma 3**) $0 \circ f = (0 - 0) \circ f = 0 \circ f - 0 \circ f = 0$, and dually. □

**Lemma 4:** An additive category has a zero object.

*Proof.* (**Lemma 4**) We have by definition an initial object $\star$. We show that it is also terminal. First, $\mathrm{id}_0$ has to be the identity of the only element in the group $\mathcal{C}(\star, \star)$. Consider the zero morphism of $\mathcal{C}(c, \star)$, and take any $f : c \to \star$. Then $f = \mathrm{id}_\star \circ f = 0 \circ f = 0$. Therefore, $f$ is also the zero morphism of $\mathcal{C}(c, \star)$, which proves uniqueness, hence that $\star$ is terminal. □

**Proposition 5:** Let $\mathcal{C}$ be additive. Then any finite coproduct is also a finite product.

*Proof.* (**Proposition 5**) Take a binary coproduct $c \sqcup d$, and consider the following diagram:




We will show that $(c \sqcup d, (\mathrm{id}_c, 0), (0, \mathrm{id}_d))$ is a product. Consider $f : a \to c$ and $g : a \to d$ and call $\varphi := \iota_c f + \iota_d g : a \to c \sqcup d$. We compute:

$$(\mathrm{id}_c, 0) \circ (\iota_c f + \iota_d g) = (\mathrm{id}_c, 0)\iota_c f + (\mathrm{id}_c, 0)\iota_d g = \mathrm{id}_c f + 0g = f,$$

and dually

$$(0, \mathrm{id}_d) \circ (\iota_c f + \iota_d g) = g.$$

Therefore, the map $\varphi$ is a morphism of cone. Observe that

$$\iota_c(\mathrm{id}_c, 0) + \iota_d(0, \mathrm{id}_d) = \mathrm{id}_{c \sqcup d},$$

as $(\iota_c(\mathrm{id}_c, 0) + \iota_d(0, \mathrm{id}_d)) \circ \iota_c = \iota_c \mathrm{id}_c + \iota_d 0 = \iota_c$, and similarly for $\iota_d$. Thus by universal property of the coproduct, it has to be the identity. Now suppose we have $\varphi' : a \to c \sqcup d$ commuting with the projections. Then:

$$\varphi' = \mathrm{id}_{c \sqcup d} \circ \varphi' = (\iota_c(\mathrm{id}_c, 0) + \iota_d(0, \mathrm{id}_d)) \circ \varphi' = \iota_c(\mathrm{id}_c, 0)\varphi' + \iota_d(0, \mathrm{id}_d)\varphi' = \iota_c f + \iota_d g = \varphi.$$

Finally, we conclude the proof by induction for arbitrary finite products. □

**Definition 6:** Let $\mathcal{C}$ be an additive category, and $f : c \to d$. The *kernel* of $f$, if it exists, is $\ker(f)$ the pullback of $f$ along the unique morphism $0 \to d$. Dually, the *cokernel* $\mathrm{coker}(f)$ is the pushout of $f$ along the unique morphism $c \to 0$.

**Definition 7:** An *abelian category* $\mathcal{C}$ is an additive category such that:

1. Every morphism admits a kernel and a cokernel.
2. Every mono is a kernel, and every epi is a cokernel.

**Remark 8:** The category **Ab** is indeed abelian. It is customary to write $X/Y$ for the cokernel of a monomorphism $Y \hookrightarrow X$. This recovers the notion of quotient group in **Ab**.

**Proposition 9:** An additive category with all kernels and cokernels is abelian if and only if for all $f$, the canonical map

$$\mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$$

is an isomorphism.

**Definition 10:** If $f : c \to d$ is a morphism in an abelian category, we define its *image*, $\mathrm{im}(f)$, to be the kernel of its cokernel. According to Proposition 9, we could also have chosen the cokernel of its kernel.

**Lemma 11:** In an abelian category, $f : c \to d$ is mono if and only if $\ker(f) \simeq 0$, and it is epi if and only if $\mathrm{im}(f) \simeq d$.

*Proof.* (**Lemma 11**) Suppose $f$ is mono, then as pullback preserves monos, we have $\iota : \ker(f) \hookrightarrow 0$, thus $\ker(f) \simeq 0$. Indeed, if we have an inclusion into a zero object, then it is an isomorphism. To prove this, consider a mono $\iota : a \hookrightarrow 0$. Next for any other $a'$, we have the zero morphisms $0_{a',a} : a' \to a$. If we have two morphisms $h, k : a \to a'$, then $\iota h = \iota k$ as 0 is terminal, hence $h = k$, as $\iota$ is mono. Conversely, suppose $\ker(f) \simeq 0$, and take $h, k : c' \to c$ equalized by $f$. Then $f \circ (h - k) = 0_{c',d}$, thus via universal property of the pullback, $h - k$ factors trough $c' \to \ker(f) \simeq 0 \to c$, hence is the zero morphism, thus $h = k$.

Dually, $f$ is epi if and only if $\mathrm{coker}(f) \simeq 0$. Thus suppose $\mathrm{coker}(f) \simeq 0$, then $\mathrm{im}(f) = \ker(\mathrm{coker}(f)) = d$, as the pullback of a map along itself is the identity. Conversely, suppose $\mathrm{im}(f) = d$, then we have the following pushout:

$$\begin{array}{ccc} \ker(f) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ c & \xrightarrow[f]{} & d \end{array}$$

which proves, via universal property, that $f$ is epi. □

**Remark 12:** Therefore, in an abelian category, every morphism $f : c \to d$ factorises uniquely as an epi followed by a mono:

$$c \stackrel{p}{\twoheadrightarrow} \mathrm{im}(f) \stackrel{\iota}{\hookrightarrow} d.$$

**Lemma 13:** Suppose we have a mono $f : c \to d$, then $\ker(d \twoheadrightarrow d/c) \simeq c$.

*Proof.* (**Lemma 13**) We compute:

$$\ker(\mathrm{coker}(f)) \simeq \mathrm{coker}(\ker(f)) \simeq \mathrm{coker}(0_c) \simeq c.$$

where the first isomorphism uses Proposition 9, and the second Lemma 11. □

**Definition 14:** In an abelian category, a sequence

$$\dots \to c_{n-1} \xrightarrow{f_n} c_n \xrightarrow{f_{n+1}} c_{n+1} \to \dots$$

is *long exact* if for all $n$, we have $\mathrm{im}(f_n) \simeq \ker(f_{n+1})$. A sequence is *short exact* if only three consecutive $c_n$ are non zero, and we write simply:

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0.$$

**Proposition 15:** In an abelian category, the following are equivalent.

1. $0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$ is short exact.

2. $f$ is mono, $g$ is epi, and $\mathrm{im}(f) = \ker(g)$.

*Proof.* (**Proposition 15**) The condition $\mathrm{im}(f) = \ker(g)$ is equivalent to exactness at $b$, and Lemma 11 proves exactness at $a$ and $c$. Indeed $\mathrm{im}(0 \to a) = 0$, which is $\ker(f)$ if and only if $f$ is mono, and $\ker(c \to 0) = c$ which is $\mathrm{im}(g)$ if and only if $g$ is epi. □

## 2.2 Exactness of functors

**Definition 16:** A functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* if it preserves finite limits, and *right exact* if it preserves finite colimits.

**Lemma 17:** A functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is:

1. Left exact if and only if it preserves (finite) direct sums and kernels;
2. Right exact if and only if it preserves (finite) direct sums and cokernels.

*Proof.* (**Lemma 17**) It is a consequence of the following facts:

1. finite (co)limits can be computed from finite (co)products and (co)equalizers;
2. direct sums of abelian groups are coproducts which are products;
3. the (co)equalizer of $f$ and $g$ is the (co)kernel of $f - g$.

□

**Proposition 18:** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories. For all exact sequences

$$0 \to a \to b \to c \to 0,$$

in $\mathcal{C}$, we have

- $F$ is left exact if and only if for all short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{C}$, the sequence $0 \to F(a) \to F(b) \to F(c)$ is exact in $\mathcal{D}$.
- $F$ is right exact if and only if for all short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{C}$, the sequence $F(a) \to F(b) \to F(c) \to 0$ is exact in $\mathcal{D}$.
- $F$ is exact if and only if for all short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{C}$, the sequence $0 \to F(a) \to F(b) \to F(c) \to 0$ is exact in $\mathcal{D}$.

*Proof.* (**Proposition 18**) Suppose $F$ is left exact. Since

$$0 \to a \xrightarrow{\iota} b \overset{p}{\twoheadrightarrow} c \to 0,$$

is short exact, we have that $0 \to a$ is the kernel of $\iota$, and $\iota$ is the kernel of $p$. Therefore, by Lemma 17

$$0 \to F(a) \xrightarrow{F(\iota)} F(b) \xrightarrow{F(p)} F(c),$$

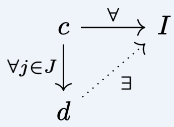
is again exact. However, to conclude that the full sequence is exact, we would need that $F$ preserves the cokernel $c \to 0$. Conversely, for the preservation of kernel, choose $f : c \to d$, and factor it as $c \xrightarrow{f'} \mathrm{im}(f) \to d$. Then $\ker(f) = \ker(f')$ and $0 \to \ker(f) \to c \to \mathrm{im}(f) \to 0$ is exact, hence so is $0 \to F(\ker(f)) \to F(c) \to F(\mathrm{im}(f))$, and thus $F(\ker(f))$ is the kernel of $F(f)$. For the coproduct, notice that $0 \to a \to a \sqcup b \to b \to 0$ is exact in any abelian category, thus we have two exact rows in:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(a) & \longrightarrow & F(a) \sqcup F(b) & \longrightarrow & F(b) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & F(a) & \longrightarrow & F(a \sqcup b) & \longrightarrow & F(b) & &
\end{array}$$

and the snake lemma shows that $F(a) \sqcup F(b) \to F(a \sqcup b)$ is an isomorphism. □

## 2.3 Injective objects

**Definition 19:** Let $\mathcal{C}$ be a category, and $J$ a collection of morphism of $\mathcal{C}$. A *$J$-injective* object $I$ of $\mathcal{C}$ is an object with the right lifting property against $J$, that is for all $j : c \to d \in J$, and all morphism $c \to I$, we have a dashed arrow:




If $J$ is the class of monomorphisms of $\mathcal{C}$, we simply say that $I$ is injective. The dual construction is called *projective*.

**Proposition 20:** Let $\mathcal{C}$ be an abelian category. The following are equivalent:

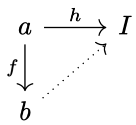
1. $I$ is injective.
2. The hom functor $\mathcal{C}(-, I) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ is exact, that is, it preserves limits and colimits.
3. For all exact sequence $0 \to c \xrightarrow{f} d$, and for all $k : c \to I$, there exists $h : d \to I$ such that $hf = k$.

Then an object $I$ is injective if and only if $\mathcal{C}(-, I)$ is exact, that is, it preserves limits and colimits.

*Proof.* (**Proposition 20**) Suppose $I$ is injective. In general $\mathcal{C}(-, I)$ preserves limits, so it is left exact. Consider in $\mathcal{C}$ an exact sequence $0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$. We want to show that:

$$\mathcal{C}(c, I) \to \mathcal{C}(b, I) \to C(a, I) \to 0$$

is exact. Take any map $h : a \to I$, then as $f$ is mono, the lifting problem:




admits a solution. Thus, $\mathcal{C}(b, I) \to \mathcal{C}(a, I)$ is surjective. The fact that $\mathcal{C}(c, I)$ is the kernel of $\mathcal{C}(b, I) \to \mathcal{C}(a, I)$ is the direct consequence of the universal property of the kernel, and does not rely on $I$ being injective.

Next, suppose $\mathcal{C}(-, I)$ is exact, and take $0 \to c \xrightarrow{f} d$ an exact sequence, and $k : c \to I$ a morphism. Then $\mathcal{C}(d, I) \to \mathcal{C}(c, I) \to 0$ is exact, thus $\mathcal{C}(d, I) \to \mathcal{C}(c, I)$ is epi, and we can find a preimage $h : d \to I$ to $k$ such that $hf = k$. Finally, notice that this condition is also precisely saying that $I$ is injective, as exact sequences $0 \to c \to d$ are exactly monomorphisms $c \to d$. □

**Definition 21:** We say that a category $\mathcal{C}$ has *enough injectives* if every object admits a monomorphism into an injective object.

**Proposition 22:** The category $\mathbf{Ab}$ has enough injectives. They are given by divisible groups, that is abelian groups $G$ such that $nG = G$, for all strictly positive integer $n$.

*Proof.* (**Proposition 22**) This proof is adapted from `https://stacks.math.columbia.edu/tag/01D6`. Take $I$ an injective group, and suppose it is not divisible by $n$, that is we have some $x \in I$ such that there is no $y$ with $ny = x$. Now consider the map $f : \mathbf{Z} \to I$ sending $m \to mx$. Then an extension along the embedding $\mathbf{Z} \hookrightarrow \frac{1}{n}\mathbf{Z}$ would be such that $nf(\frac{1}{n}) = f(1) = x$, contradiction.

Conversely, consider $A \subseteq B$ two abelian groups, $I$ a divisible group, and $\varphi : A \to I$ any morphism. We will apply Zorn's lemma. Thus, consider the set of all morphisms $\varphi' : A' \to I$ such that $A \subseteq A' \subseteq B$ and $\varphi'$ restricts to $\varphi$ on $A$. Define the partial order $(A', \varphi') \geq (A'', \varphi'')$ if and only if $A'' \subseteq A'$, and $\varphi'$ restricts to $\varphi''$ on $A''$. If we have an ordered collection $\{(A_k, \varphi_k)\}_{k \in K}$ of such pairs, then taking $(\cup_k A_k, \tilde{\varphi})$ where $\tilde{\varphi}(a_k) = \varphi_k(a_k)$ is well defined by the restrictions, is a maximal element. Thus, Zorn's lemma applies, and we get a maximal pair $(A', \varphi')$. It suffices then to show that in that case $A' = B$. By contradiction, take $x \in B$ and $x \notin A'$.

Suppose first that there exists no $n \in \mathbb{N}$ such that $nx \in A'$. Then $A' \oplus \mathbf{Z} \simeq A' + \mathbf{Z}x \subseteq B$, and $\varphi'$ can be extended to $A' + \mathbf{Z}x$ by sending $x$ to the identity of $I$, which contradicts maximality of

$(A', \varphi')$. Otherwise, we take $n$ minimal (strictly positive) such that $nx \in A'$. As $I$ is divisible, we have some $z \in I$ such that $nz = \varphi'(nx)$. Notice that if $mx \in A'$, then $m = kn$ for an integer $k$, otherwise $[m \mod n] \cdot x$ would be in $A'$, contradicting minimality. Thus, for the contradiction, we extend $\varphi'$ to $A' + \mathbf{Z}x$ by sending $a + mx$ to $\varphi'(a) + mz$. This definition makes sense, as if $mx \in A'$, then $\varphi(a + mx) = \varphi(a + knx) = \varphi'(a) + k\varphi'(nx) = \varphi'(a) + mz$.

Now that it is established that injective objects are divisible groups, we will embed any abelian group $A$ into a direct product of $\mathbf{Q}/\mathbf{Z}$. We state without proving that a product of injective groups is injective, and that $\mathbf{Q}/\mathbf{Z}$ is itself injective. We create by universal property a monomorphism:

$$\beta : A \to \prod_{a \in A, a \neq 0} \mathbf{Q}/\mathbf{Z},$$

which on $a \in A, a \neq 0$ is given by $\beta_a$ that we define as follows. First call $(a)$ is the subgroup generated by $a$, and create the map $(a) \to \mathbf{Q}/\mathbf{Z}$ sending $a$ to an arbitrary non-zero element if the order of $a$ is infinite, and to, say, $1/n$ if $a$ has order $n$. As $\mathbf{Q}/\mathbf{Z}$ is injective, we can solve the lifting problem:

$$\begin{array}{ccc} (a) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\ \downarrow & \nearrow & \\ A & & \end{array}$$

with $\beta_a$, which is a monomorphism. □

We now wish to prove that sheaves on a Grothendieck topos have enough injectives. We consider the neat proof from [Joh14].

**Theorem 23 (Barr's theorem):** If $\mathcal{E}$ is a Grothendieck topos, then there is a surjective geometric morphism

$$\mathcal{F} \to \mathcal{E}$$

where $\mathcal{F}$ satisfies the axiom of choice.

For a Grothendieck topos $\mathcal{E}$, call $\mathbf{Ab}(\mathcal{E})$ the sheaves with values in $\mathbf{Ab}$.

**Lemma 24 ([Joh14, 8.12]):** If $f : \mathcal{E}' \to \mathcal{E}$ is a geometric morphism, then

1. The direct image $f_* : \mathbf{Ab}(\mathcal{E}') \to \mathbf{Ab}(\mathcal{E})$ preserves injectives.
2. If $f$ is a surjection, and $\mathbf{Ab}(\mathcal{E}')$ has enough injective, so has $\mathbf{Ab}(\mathcal{E})$.

*Proof.* (Lemma 24) Suppose we have an injective object $e \in \mathcal{E}'$, then consider any diagram

$$\begin{array}{ccc} a & \longrightarrow & f_*(e) \\ \downarrow & & \\ b & & \end{array}$$

with $a \to b$ mono, and we transpose:

$$\begin{array}{ccc} f^*(a) & \longrightarrow & e \\ \downarrow & & \\ f^*(b) & & \end{array}$$

Again, $f^*(a) \to f^*(b)$ is mono, as the inverse image $f^*$ preserve finite limits. Therefore, we can find a lift in $\mathcal{E}'$, and transpose again.

Next, suppose $f$ is surjective. Take an abelian sheaf $a \in \mathcal{E}$, and embed it with $f^*(a) \hookrightarrow e$ with $e$ injective in $\mathcal{E}'$. Then we consider:

$$a \xrightarrow{\eta_a} f_* f^*(a) \to f_*(e).$$

This time, $f_* f^*(a) \to f_*(e)$ is mono because $f_*$ is a right adjoint, so preserves finite limits. Then, the unit of a geometric morphism is by definition mono whenever it is surjective. □

Finally, we state the main result:

**Theorem 25:** Let $\mathcal{E}$ be a Grothendieck topos. Then $\mathbf{Ab}(\mathcal{E})$ has enough injectives.

*Proof.* (**Theorem 25**) Using Theorem 23, it is enough to consider the case when the Grothendieck topos satisfies the axiom of choice, but then we can exactly mimic internally the proof Proposition 22, that rely on Zorn's lemma, that is the axiom of choice, and prove that an abelian sheaf is divisible if and only if is is a divisible abelian group at each point. Then, we sheafify the constant abelian presheaf $\mathbf{Q}/\mathbf{Z}$, and prove that any abelian sheaf can be embed into its direct products, pointwise computed in a sheaf topos. □

**Remark 26:** We can also compute more explicitly an embedding $X \hookrightarrow I$ into an injective abelian sheaf. See `https://stacks.math.columbia.edu/tag/01DL`, which in fact rely on the same result: we first prove that **Ab** has enough injectives, and we use them pointwise to prove that abelian *presheaves* have enough injective. Then after a transfinite recursion, we conclude that we can "sheafify" this result.

# 3 Sheaf cohomology

As explained in the introduction, the outline of sheaf cohomology is that given an abelian sheaf $X \in \mathbf{Ab}(\mathcal{E})$, we can consider an injective resolution, that is an exact sequence:

$$0 \to X \to I^0 \to I^1 \to \dots,$$

where each $I^n$ is an injective object. Then, given a left exact functor $\Gamma : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$, we have the sequence:

$$0 \to \Gamma(I^0) \to \Gamma(I^1) \to \Gamma(I^2) \to \dots$$

obtained by applying $\Gamma$ and composing $0 \to \Gamma(X) \to \Gamma(I^0)$ (which is already exact by Proposition 18). The problem is that, $\Gamma$ might lack some right-exactness, and thus the sequence might not be exact. This default of exactness is measured precisely by the quotients:

$$\frac{\ker(\Gamma(I^q) \to \Gamma(I^{q+1}))}{\operatorname{im}(\Gamma(I^{q-1}) \to \Gamma(I^q))}$$

If $\Gamma$ is the global section functor, that is $\Gamma = \mathcal{E}(1, -) : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$, then this quotient is $H^q(\mathcal{E}, X)$, the $q$th cohomology group of $\mathcal{E}$ with value in $X$.

## 3.1 Injective resolution

Let $\mathbb{A}$ be an abelian category.

**Definition 27:** A *cochain complex* $(C^\bullet, \delta)$ in $\mathbb{A}$ is for all *non negative* integer $k$, an object $C^k$ of $\mathbb{A}$, together with maps, called *differentials*,

$$\delta^k : C^k \to C^{k+1}$$

such that $\delta \circ \delta = 0$. A morphism of cochain complex $f : (C^\bullet, \delta) \to (D^\bullet, \delta')$ is the data of morphisms $f : C^k \to D^k$ in each degree commuting with the differentials. This forms the category $\mathrm{Ch}^\bullet(\mathbb{A})$.

**Remark 28:** It is often useful to extend a chain complex on the left by letting $C^k = 0$ for negative values of $k$, with trivial differentials.

**Definition 29:** The *qth cohomology group* of a cochain complex $(C^\bullet, \delta)$ is the abelian object:

$$H^q(C^\bullet) := \frac{\ker(\delta^q)}{\operatorname{im}(\delta^{q-1})}$$

As maps of cochain complexes commute with differentials, this define a functor $H^q : \mathrm{Ch}^\bullet(\mathbb{A}) \to \mathbb{A}$.

**Definition 30:** A morphism of cochain complex $f : C^\bullet \to D^\bullet$ is called a *quasi-isomorphism* if it induces an isomorphism

$$H^q(f) : H^q(C^\bullet) \simeq H^q(D^\bullet)$$

on each cohomology group.

Finally, we arrive to our definition of interest.

**Definition 31:** Let $X \in \mathbb{A}$. An *injective resolution* of $X$, is a cochain complex $I^\bullet \in \mathrm{Ch}^\bullet(\mathbb{A})$ such that for all $k$, $I^k$ is an injective object, together with a quasi-isomorphism $i : X \to I^\bullet$, where $X$ is identified with the complex:

$$X \to 0 \to 0 \to \dots$$

**Remark 32:** There is a model structure on $\mathrm{Ch}^\bullet(\mathbb{A})$ where cofibrations are monomorphisms (on positive degree), fibrations are epimorphisms with injective kernels, and weak equivalences are quasi-isomorphisms. We see that an injective complex $I^\bullet$ is then a fibrant object, and an injective resolution is precisely a fibrant replacement, that is, we replace a complex with another one, more suited to do homotopy theory.

**Proposition 33:** An injective resolution of $X$ is equivalently the data of an exact sequence

$$0 \to X \to I^0 \to I^1 \to \dots$$

where $I^k$ is injective.

*Proof.* (**Proposition 33**) Suppose we have an injective resolution

$$\begin{array}{ccccccc} X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ \downarrow{\scriptstyle i_0} & & \downarrow & & \downarrow & & \\ I^0 & \xrightarrow[\delta^0]{} & I^1 & \xrightarrow[\delta^1]{} & I^2 & \xrightarrow[\delta^2]{} & \dots \end{array}$$

then $H^q(X) = X$ if $k = 0$, and $H^q(X) = 0$ for $q > 1$. Then, for $q > 0$, the induced isomorphism means that $H^{q+1}(I^\bullet) = 0$, that is $\ker(\delta^{q+1}) = \operatorname{im}(\delta^q)$, proving exactness at $q > 0$. For exactness at $I^0$, notice that $i_0$ induces an isomorphism $H^0(X) = X \simeq \ker(\delta^0) = H^0(I^\bullet)$, which also prove that $i_0$ is injective, as it is the kernel of $\delta^0$. The converse uses the same argument backward. □

**Remark 34:** Thus, we will write $0 \to X \to I^\bullet$ for an injective resolution of $X$.

**Lemma 35:** Suppose $\mathbb{A}$ has enough injectives, then any $X \in \mathbb{A}$ has an injective resolution.

*Proof.* (**Lemma 35**) For the sake of notation, call $I^{-1} := X$. We construct an exact sequence by induction. For the base case, take any mono $\delta^{-1} = i : X \hookrightarrow I^0$ into an injective object, which exists because $\mathbb{A}$ has enough injectives. Then $0 \to X \to I^0$ is indeed exact. Suppose we constructed $\delta^k : I^k \to I^{k+1}$ for $0 \leq k < q$. Then consider any mono $I^{q-1}/\operatorname{im}(\delta^{q-2}) \hookrightarrow I^q$ into an injective object $I^q$. We define the next term $\delta^q$ to be the induced map:

$$I^{q-1} \twoheadrightarrow I^{q-1}/\operatorname{im}(\delta^{q-2}) \hookrightarrow I^q.$$

Then according to Lemma 11 and Lemma 13, $\ker(\delta^q) = \ker(I^{q-1} \twoheadrightarrow I^{q-1}/\operatorname{im}(\delta^{q-2})) = \operatorname{im}(\delta^{q-2})$, so the sequence is exact at $I^{q-1}$, and we can continue the induction. We therefore end up with an exact sequence:

$$0 \to X \to I^0 \to I^1 \to \dots$$

made of injective objects, which is, according to Proposition 33, the same thing as an injective resolution. □

## 3.2 Homotopy of complexes

We prove some homotopy independence result of injective resolution, that will be useful to define sheaf cohomology.

**Definition 36:** Two map of complexes $f, g : A^\bullet \to B^\bullet$ are said to be *homotopic* if for all $K \geq 0$, there are morphisms $\varphi^k : A^k \to B^{k-1}$ such that $f^k - g^k = \delta^{k-1} \circ \varphi^k + \varphi^{k+1} \circ \delta^k$. We write $f \simeq g$. In a diagram, we have maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \dots \\ & & \downarrow\downarrow & \overset{\varphi^1}{\swarrow} & \downarrow\downarrow & \overset{\varphi^2}{\swarrow} & \downarrow\downarrow & & \\ 0 & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & \dots \end{array}$$

where the sum of the two colored triangles equals the difference of the two parallel maps between them. We say that $f$ is *null-homotopic* if it is homotopic to the zero map $0 : A^\bullet \to B^\bullet$. Thus, to say that $f$ and $g$ are homotopic is to say that $f - g$ is null-homotopic. Two complexes $A^\bullet, B^\bullet$ are *homotopic* whenever there are two maps $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ such that $g \circ f \simeq \operatorname{id}_{A^\bullet}$ and $f \circ g \simeq \operatorname{id}_{B^\bullet}$.

**Lemma 37:** If two maps of complex are homotopic, then they induce the same cohomology maps.

*Proof.* (**Lemma 37**) Suppose $f : A^\bullet \to B^\bullet$ is null homotopic, and take any $x \in H^q(A^\bullet)$, so by definition $\delta^{q+1}(x) = 0$. Then $f(x) = \delta^{q-1}\varphi(x) + \varphi\delta^{q+1}(x) = \delta^{q-1}\varphi(x) \in \operatorname{im}(\delta^{q-1})$ thus is zero in $H^q(B^\bullet)$. Hence, $f$ is the zero map. Now, if $f$ and $g$ are homotopic, then $f - g$ is the zero map on the cohomology groups, thus $f$ and $g$ are the same map in cohomology. □

**Corollary 38:** If two complexes are homotopic, they have the same cohomology.

*Proof.* (**Corollary 38**) Suppose $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ is a homotopy of complex. Then by 37, the maps $H^\bullet(f)$ and $H^\bullet(g)$ compose to the identity $H^\bullet(\mathrm{id})$ both ways, which proves the isomorphism of cohomology. □

The goal is to prove that two injective resolutions are homotopic.

**Theorem 39:** Let $0 \to X \to I^\bullet$ and $0 \to X \to J^\bullet$ be two injective resolutions of $X$, then $I^\bullet$ and $J^\bullet$ are homotopic.

### 3.3 Cohomology

Finally, we define sheaf cohomology inside a general Grothendieck topos, and show that it is independent of the injective resolution we chose. This last point in fact corresponds to the fact that the cohomology is really computed in the homotopy category of cochain complexes, and thus is independent of the choice of fibrant replacement, see Remark 32, but these considerations are out of scope of this note. Here, it will simply follow from the homotopic considerations of the previous section.

We fix a Grothendieck topos $\mathcal{E}$, and even though the result can be obtained more general with an abelian category $\mathbb{A}$ with enough injectives, we stick to $\mathbb{A} = \mathbf{Ab}$, and consider the category of abelian sheaves $\mathbf{Ab}(\mathcal{E})$. We use the definition from [Joh14].

**Definition 40:** Let $X \in \mathbf{Ab}(\mathcal{E})$. We call $\Gamma_X$ the functor $\mathcal{E}(X, -) : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$. If $X \in \mathcal{E}$, we may also consider $\Gamma_X$ by composing with the free abelian group functor.

**Lemma 41:** The functor $\Gamma_X$ has a left adjoint, so is left-exact.

*Proof.* (**Lemma 41**) Call $a \dashv i$ the (enriched) adjoint pair sheafification-forgetful. The proof is inspired from this post. We recall without proof that we have a tensor product of abelian groups $A \otimes B$ such that

$$A \otimes - \dashv \mathbf{Ab}(A, -),$$

and that this passes to abelian presheaves, that is, for a category $\mathcal{C}$, a natural isomorphism

$$\mathbf{Ab}(\hat{\mathcal{C}})(A \otimes X, Y) \simeq \mathbf{Ab}(A, \mathbf{Ab}(\hat{\mathcal{C}})(X, Y)),$$

where $A \otimes X$ is computed pointwise.

Call $\mathcal{C}$ a site of $\mathcal{E}$. Let $A$ be an abelian group, and $X \in \mathbf{Ab}(\mathcal{E})$, we define the sheaf $A \cdot X$ by:

$$(A \cdot X)(c) = a(A \otimes iF(c)).$$

We show that the tensoring functor $\Delta_X : \mathbf{Ab} \to \mathbf{Ab}(\mathcal{E})$ sending $A$ to $A \cdot X$ is the desired left adjoint. We have the natural isomorphisms:

$$\begin{aligned}
\mathbf{Ab}(\mathcal{E})(A \cdot X, Y) &\simeq \mathbf{Ab}(\mathcal{E})(a(A \otimes iF), Y) \\
&\simeq \mathbf{Ab}(\hat{\mathcal{C}})(A \otimes iX, iY) \\
&\simeq \mathbf{Ab}(A, \mathbf{Ab}(\hat{\mathcal{C}})(iX, iY)) \\
&\simeq \mathbf{Ab}(A, \mathbf{Ab}(\mathcal{E})(aiX, Y)) \\
&\simeq \mathbf{Ab}(A, \mathbf{Ab}(\mathcal{E})(X, Y)).
\end{aligned}$$

□

**Construction 42:** Let $U \in \mathcal{E}$. Let $F \in \mathbf{Ab}(\mathcal{E})$. The *qth cohomology group* of the space $U$ with value in $F$ is the $q$th cohomology group of the cochain complex $\Gamma_U(I^\bullet)$ for any injective resolution of $F$. More explicitly, if $0 \to U \to I^\bullet$ is an injective resolution, then define the $q$th cohomology group to be:

$$H^q(U, F) := \frac{\ker(\Gamma_U(I^q) \to \Gamma_U(I^{q+1}))}{\operatorname{im}(\Gamma_U(I^{q-1}) \to \Gamma_U(I^q))},$$

with convention that $I^{-1} = 0$.

*Proof.* (**Construction 42**) For this to be well defined, we need to show that it does not depend on the injective resolution we chose. We know by Theorem 39 that any two injectives resolutions of $X$ are homotopic, and so is there image trough $\Gamma$ (which is an additive functor), thus according to Corollary 38, they have isomorphic cohomology groups. □

**Remark 43:** If $\mathcal{E}$ is a category of sheaves on a topological space $X$, then $X$ as a sheaf is the terminal object, thus $\Gamma_X$ is indeed the global section functor, and we recover the usual definition of cohomology.

## References

[Joh14] P. T. Johnstone. *Topos theory.* Dover books on mathematics. Dover Publications, Inc, Mineola, New York, dover edition edition, 2014.