What is the Yoneda Lemma?

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I started studying category for almost a year now, and one of the fundamental result that still strikes me is the Yoneda Lemma. I know its statement, I use it everyday in my work, but somehow I cannot build any kind of intuition about it. I blame the notion of presheaf that is not yet intuitive to me. Here is a paper where I try to create a detailed understanding of this result and its consequences.

We assume the reader is familiar with the basic notions of category theory (categories, functors, natural transformations). If not, we encourage the reader to go through the first chapter of Emily Riehl's Category Theory in Context (from which the example about preorder in this paper is taken).

The first section is dedicated to the notion of presheaf, we present the basics of it and some intuition behind it. In the second part, we move to the Yoneda Lemma, and discuss the density formula through the example of graphs. Note that all of this is made from scratch, and no higher level tools, like co-end, are used. In future work, it can be interesting, to repackage all this information using them.

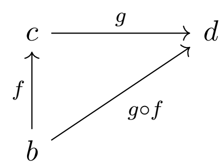
1 Presheaves

1.1 How to be a presheaf

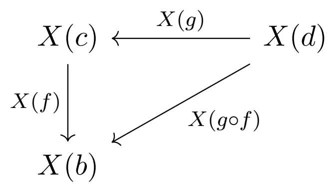
Let $\mathcal{C}$ be a locally small category. A *presheaf* is a functor $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$. This is the data of:

- a set $X(c)$, for each object $c \in \mathcal{C}$
- a set-theoretic map $X(f) : X(d) \to X(c)$, for each morphism $f : c \to d$

This construction is functorial. It means that when we have



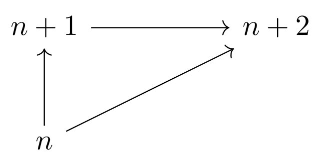
then



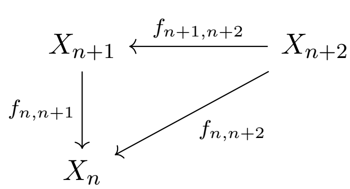
commutes. Moreover, for all $d \in \mathcal{C}$, $X(\mathrm{id}_d) = \mathrm{id}_{X(d)}$. In the case where the category $\mathcal{C}$ is discrete (no morphism besides identities), a presheaf is just a bunch of indexed sets, one for each $c \in \mathcal{C}$. So a good intuition for presheaves is a generalization of indexed sets. Let's strengthen this intuition to the case where the category is a preorder $\mathcal{N}$. To simplify further, suppose $\mathcal{N}$ has objects the natural numbers (and so a unique morphism $n \to m$ whenever $n \leq m$). A presheaf $X : \mathcal{N}^{\mathrm{op}} \to \mathrm{Sets}$ is

- a collection of sets $X_0, X_1, X_2, \ldots$, for each natural number $n \in \mathcal{N}$
- a map $f_{n,m} : X_m \to X_n$, whenever $n \leq m$,

Is that it? No, we have the functoriality of the maps too. Let us write it in the special case $n \leq n+1 \leq n+2$, for a chosen $n \in \mathcal{N}$. We have



meaning that we have



that is

$$f_{n,n+2} = f_{n,n+1} \circ f_{n+1,n+2}$$

More generally, we can prove that for $p \geq 0$, we have

$$f_{n,n+p} = f_{n,n+1} \circ f_{n+1,n+2} \circ \cdots \circ f_{n+(p-1),n+p}$$

What does that tell us? That to specify our presheaf X, it suffices to give the maps $f_{n,n+1} : X_{n+1} \to X_n$, for all $n \geq 0$, and the rest of the maps will follow from the previous decomposition. Thus, the presheaf X is indeed a bunch of indexed sets, but they are also realted in a coherent manner with the maps $f_{n,n+1}$. For instance, suppose all the $f_{n,n+1}$ are injections. Then a presheaf X can be seen as a family $(X_n)_{n\in\mathbb{N}}$ of decreasing sets (i.e. $X_n \supseteq X_m$ whenever $n \leq m$). Likewise, a presheaf over $\mathcal{N}^{\mathrm{op}}$ with injective maps would be a family $(X_n)_{n\in\mathbb{N}}$ of increasing sets

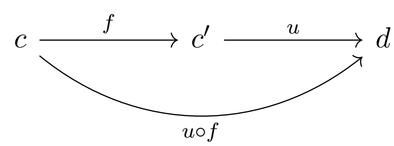
Back to the general category $\mathcal{C}$. It is locally small ($\mathcal{C}(c, d)$ is a set for any objects $c, d \in \mathcal{C}$), so for every $d \in \mathcal{C}$, we have the contravariant hom functor $h_d : \mathcal{C}^{\mathrm{op}} \to \text{Sets}$ defined as follow.

$$\begin{aligned} h_d : \mathcal{C}^{\mathrm{op}} &\to \text{Sets} \\ c &\mapsto \mathcal{C}(c, d) \\ f : c' \to c &\mapsto \text{-} \circ f \end{aligned}$$

To be more explicit, if $f : c \to c'$ in $\mathcal{C}$, then we can define the post-composition function:

$$\begin{aligned} \text{-} \circ f : \mathcal{C}(c', d) &\to \mathcal{C}(c, d) \\ u &\mapsto u \circ f \end{aligned}$$

as in the following diagram.



This construction is functorial, this is by associativity of composition.

$$(u \circ f) \circ g = u \circ (f \circ g)$$

The left hand side is $h_d(g) \circ h_d(f)(u)$ and the right hand side is $h_d(f \circ g)(u)$.

The hom functors are a fundamental in category theory, and the Yoneda Lemma is all about them. One of its consequences is that any presheaves is the gluing of such hom functors, in the same flavor as the rational numbers are *dense* in the real numbers, the class of hom functors $\{h_d : \mathcal{C}^{\mathrm{op}} \to \text{Sets} \mid d \in \mathcal{C}\}$ is also dense in the presheaves over $\mathcal{C}$. We will see how to prove such a statement later.

Let us compute the hom functor in the case of the preorder $\mathcal{N}$. We fix an $m \in \mathcal{N}$. We have

$$h_m(n) = \mathcal{N}(n, m) = \begin{cases} \{\star\} & \text{if } n \leq m \\ \emptyset & \text{if } n > m \end{cases}$$

In order to understand the maps, let us do a very quick recap about set theory.

- There is exactly one function $\emptyset_X : \emptyset \to X$ for any set X (including $X = \emptyset$), called the empty map, whose graph is $\emptyset$
- There is exactly one function $\star_X : X \to \{\star\}$ for any set X, whose graph is $\{(x, \star) \mid x \in X\}$ (hence the graph is $\emptyset$ whenever $X = \emptyset$, and so $\emptyset_\star = \star_\emptyset$).

As we saw previously, it suffices to understand, for every $n \in N$, where the unique map $n < n+1$ in $\mathcal{N}$ is sent by the functor h_m. Let us divide the cases, and recall that $h_m(n < n+1) : h_m(n+1) \to h_m(n)$.

- If $n < n+1 \leq m$, then $h_m(n < n+1) : \{\star\} \to \{\star\}$ is the unique map $\star_\star$ defined above
- If $m = n < n+1$, then $h_m(n < n+1) : \emptyset \to \{\star\}$ is the unique map $\emptyset_\star = \star_\emptyset$ defined above
- If $m < n < n+1$, then $h_m(n < n+1) : \emptyset \to \emptyset$ is the unique map $\emptyset_\emptyset$ defined above

1.2 Morphism of presheaves

Recall that the presheaves on a category $\mathcal{C}$ form a category where a map between $X, Y : \mathcal{C}^{\text{op}} \to \text{Sets}$ is a natural transformation $t : X \to Y$. Such a natural transformation is the data of set-theoretic maps $t_c : X(c) \to Y(c)$, for every $c \in \mathcal{C}$, such that for any arrow $f : c \to d$, the following diagram commutes.

$$\begin{array}{ccc} X(c) & \xleftarrow{\quad X(f) \quad} & X(d) \\ {\scriptstyle t_c}\big\downarrow & & \big\downarrow{\scriptstyle t_d} \\ Y(c) & \xleftarrow[\quad Y(f) \quad]{} & Y(d) \end{array}$$

That is
$$t_c \circ X(f) = Y(f) \circ t_d$$

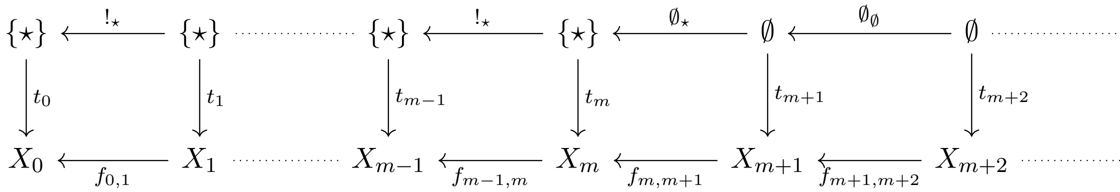
Specializing to the example of the preorder $\mathcal{N}$, suppose we have a presheaf X which sends $n \leq m$ to $f_{n,m}$ and a presheaf Y which sends $n \leq m$ to $g_{n,m}$, then a natural transformation t is the data of maps $t_n : X_n \to Y_n$ such that the following infinite diagram commutes.

$$\begin{array}{ccccccccccc} X_0 & \xleftarrow{f_{0,1}} & X_1 & \cdots\cdots & X_{n-1} & \xleftarrow{f_{n-1,n}} & X_n & \xleftarrow{f_{n,n+1}} & X_{n+1} & \cdots\cdots \\ {\scriptstyle t_0}\big\downarrow & & {\scriptstyle t_1}\big\downarrow & & {\scriptstyle t_{n-1}}\big\downarrow & & {\scriptstyle t_n}\big\downarrow & & {\scriptstyle t_{n+1}}\big\downarrow & \\ Y_0 & \xleftarrow[g_{0,1}]{} & Y_1 & \cdots\cdots & Y_{n-1} & \xleftarrow[g_{n-1,n}]{} & Y_n & \xleftarrow[g_{n,n+1}]{} & Y_{n+1} & \cdots\cdots \end{array}$$

Now, let us take any $m \in \mathcal{N}$, a presheaf $X : \mathcal{N}^{\text{op}} \to \text{Sets}$ (sending $n \leq m$ to $f_{n,m}$), and let us see what is the data needed to define a natural transformation $t : h_m \to X$. We can rewrite the diagram above and we have

$$\begin{array}{ccccccccccc} h_m(0) & \longleftarrow & h_m(1) & \cdots\cdots & h_m(m-1) & \longleftarrow & h_m(m) & \longleftarrow & h_m(m+1) & \cdots\cdots \\ \big\downarrow{\scriptstyle t_0} & & \big\downarrow{\scriptstyle t_1} & & \big\downarrow{\scriptstyle t_{m-1}} & & \big\downarrow{\scriptstyle t_m} & & \big\downarrow{\scriptstyle t_{m+1}} & \\ X_0 & \xleftarrow[f_{0,1}]{} & X_1 & \cdots\cdots & X_{m-1} & \xleftarrow[f_{m,m-1}]{} & X_m & \xleftarrow[f_{m,m+1}]{} & X_{m+1} & \cdots\cdots \end{array}$$

and we can replace the values that we already computed, like in the following.



We can see that the data of this natural transformation is already determined for $n > m$. Indeed, $t_n : \emptyset \to X_n$ has to be $\emptyset_{X_n}$, because it is the only such map. Thus, the only data needed is $t_n : \{\star\} \to X_n$ for all $n \leq m$. Recall that a function $f : \{\star\} \to X$ is the same thing as an element $x \in X$. This is because f is completely determined by $f(\star) \in X$, and each element of X determines such a function. Thus, instead of maps $t_n : \{\star\} \to X_n$, we will pick points $t_n(\star) = x_n \in X_n$ for every $n \leq m$. Is that it? To define a natural transformation $t : h_m \to X$, do we only need to pick points $x_n \in X_n$ for $n \leq m$? No we need even less. Recall that in the previous diagram, all the squares commute by naturality, hence for instance:

$$t_0 \circ !_0 = f_{0,1} \circ t_1$$

so, by applying this identity of functions to the element $\star$, we have

$$t_0 \circ !_0(\star) = f_{0,1} \circ t_1(\star)$$

that is

$$t_0(\star) = f_{0,1}(t_1(\star))$$

That means $x_0 = f_{0,1}(x_1)$. The choice of x_0 is not free, it is conditioned by the one of x_1. Similarly the choice of x_1 boils down to the choice of x_2, which reduces to the one of x_3, ..., until the choice of $x_m = t_m(\star)$. The choice of x_m reduces to nothing because the naturality condition is

$$t_m \circ \emptyset_\star = f_{m,m+1} \circ t_{m+1}$$

but the domain of this function is the empty set! Thus, we just showed that any natural transformation $t : h_m \to X$ is determined by an element $x \in X_n$. Conversely, we can apply the construction detailed above and create a natural transformation $t_x : h_m \to X$ for any $x \in X$. What we just established is the Yoneda Lemma.

Lemma 1.1 (Yoneda in $\mathcal{N}$)**.** *The natural transformations from h_m to X are in bijective correspondence with the set $X(m)$.*

In fact, there is more than that, such a correspondence is natural both in X and m (we will see more precisely what that means), and the element $x \in X_m$ associated to $t : h_m \to X$ is $t_m(\mathrm{id}_m) \in X_m$. This last fact was a little bit hidden, but true, in the previous example. We have that $h_m(m)$ is a singleton, but this element has to be the identity map, as $\mathrm{id}_m \in h_m(m)$.

This is a pretty amazing fact. A natural transformation $t : h_m \to X$ is a priori the data of an infinite number of maps t_n for $n \geq 0$. The Yoneda lemma tells us that the data of a single point in $X(m)$ suffices to determine it completely. It is even more impressive that it is true when the $h_m(n)$ are not simply $\emptyset$ or $\{\star\}$, but any sets, with why not infinite cardinals. In the next section, we will prove the Yoneda Lemma in its full generality, and see some of its impressive consequences.

2 The Yoneda Lemma

2.1 Statement and proof

We fix a locally small category $\mathcal{C}$, and we call $\hat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ the category of presheaves over $\mathcal{C}$. For any presheaf $X \in \hat{\mathcal{C}}$ and any $c \in \mathcal{C}$, the Yoneda Lemma tells us that there is a bijection between $\hat{\mathcal{C}}(h_c, X)$ and $X(c)$. That is, for every $X \in \hat{\mathcal{C}}$ and $c \in C$, we have

$$\phi_{c,X} : \hat{\mathcal{C}}(h_c, X) \simeq X(c)$$

The Yoneda Lemma also asserts that the data of $(\phi_{c,X})_{c \in \mathcal{C}, X \in \hat{\mathcal{C}}}$ can be collected into a natural transformation between two functors. Thus, not only do we have a bunch of bijections, but they also interact nicely. Let us be fully precise, and define the two functors. The first one is

$$\begin{aligned} H : \mathcal{C}^{\mathrm{op}} \times \hat{\mathcal{C}} &\to \mathrm{Sets} \\ (c, X) &\mapsto \hat{\mathcal{C}}(h_c, X) \\ (c \xrightarrow{f} d, X \xrightarrow{t} Y) &\mapsto (u \mapsto t \circ u \circ f^*) \end{aligned}$$

where $f^* = \text{-} \circ f$, the post composition previously defined. Note that f is an arrow of $\mathcal{C}^{\mathrm{op}}$, so $f^* : h_d \to h_c$, as needed. We can easily check that this construction is functorial. Note that this functor is a priori not well defined as $\hat{\mathcal{C}}(h_c, X)$ need not to be a set. Hence we postpone the well definition of this functor to the proof of the Yoneda Lemma, where the existence of the bijection guarantees it.

The second functor is

$$\begin{aligned} P : \mathcal{C}^{\mathrm{op}} \times \hat{\mathcal{C}} &\to \mathrm{Sets} \\ (c, X) &\mapsto X(c) \\ (c \xrightarrow{f} d, X \xrightarrow{t} Y) &\mapsto Y(f) \circ t_c \end{aligned}$$

Alternatively, we could have taken $P(c \xrightarrow{f} d, X \xrightarrow{t} Y) = t_d \circ X(f)$, but this is equal to $Y(f) \circ t_c$ by naturality.

In this precise setting, we can state the Yoneda Lemma.

Theorem 2.1 (Yoneda Lemma)**.** *There exists a natural isomorphism* $\phi : H \simeq P$*.*

We will do the proof in detail, starting with the bijection $\phi_{c,X} : \hat{\mathcal{C}}(h_c, X) \simeq X(c)$. Let $t : h_c \to X$, and chose any $d \in \mathcal{C}$, we want to determine $t_d : \mathcal{C}(d, c) \to X(d)$. For any $f \in \mathcal{C}(d, c)$, as t is a natural transformation, we have the following commutative square.

$$\begin{array}{ccc} \mathcal{C}(c, c) & \xrightarrow{f^*} & \mathcal{C}(d, c) \\ \downarrow{\scriptstyle t_c} & & \downarrow{\scriptstyle t_d} \\ X(c) & \xrightarrow[X(f)]{} & X(d) \end{array}$$

That means

$$t_d \circ f^* = X(f) \circ t_c : \mathcal{C}(c, c) \to X(d)$$

In particular, instantiating this identity for $id_c \in \mathcal{C}(c, c)$, we have

$$t_d \circ f^*(id_c) = X(f) \circ t_c(\mathrm{id}_c)$$

and $f^*(id_c) = id_c \circ f = f$, hence

$$t_d(f) = X(f) \circ t_c(\mathrm{id}_c)$$

What we just proved now is that

$$\begin{aligned} \phi_{c,X} : \hat{\mathcal{C}}(h_c, X) &\to X(c) \\ t &\mapsto t_c(\mathrm{id}_c) \end{aligned}$$

is injective. Indeed, if $\phi_{c,X}(t) = \phi_{c,X}(t')$, then for any $d \in C$, and $f : d \to c$, then

$$t_d(f) = X(f) \circ t_c(\mathrm{id}_c) = \phi_{c,X}(t) = \phi_{c,X}(t') = X(f) \circ t'_c(\mathrm{id}_c) = t'_d(f)$$

proving that $t = t'$. In particular, this ensures that the functor H is well defined, as $H(c, X) = \hat{\mathcal{C}}(h_c, X)$ injects into the set $X(c)$ and is therefore a set. Conversely, pick $u \in X(c)$ and define for all $d \in C$, and $f : d \to c$,

$$t_d(f) = X(f)(u)$$

To ensure that $t \in \hat{\mathcal{C}}(h_c, X)$, we want to prove that, for any $f : d \to d'$, the following square is commutative.

$$\begin{array}{ccc} \mathcal{C}(d', c) & \xrightarrow{f^*} & \mathcal{C}(d, c) \\ \downarrow{\scriptstyle t_{d'}} & & \downarrow{\scriptstyle t_d} \\ X(d') & \xrightarrow[X(f)]{} & X(d) \end{array}$$

For $g : d' \to c$,

$$t_d \circ f^*(g) = t_d(g \circ f) = X(g \circ f)(u) = (X(f) \circ X(g))(u) = X(f)(X(g)(u)) = X(f) \circ t_d(g)$$

This establishes that $\phi_{c,X}$ is a bijection. We will now prove the naturality in the two variables at the same time. Consider $f : c \to c'$ and $t : X \to Y$. We want to show that the following square commutes.

$$\begin{array}{ccc} H(c, X) & \xrightarrow{H(f,t)} & H(c', Y) \\ \downarrow{\scriptstyle \phi_{c,X}} & & \downarrow{\scriptstyle \phi_{c',Y}} \\ P(c, X) & \xrightarrow[P(f,t)]{} & P(c', Y) \end{array}$$

that is

$$\begin{array}{ccc} \hat{\mathcal{C}}(h_c, X) & \xrightarrow{t \circ - \circ f^*} & \hat{\mathcal{C}}(h_{c'}, Y) \\ \downarrow{\scriptstyle \phi_{c,X}} & & \downarrow{\scriptstyle \phi_{c',Y}} \\ X(c) & \xrightarrow[Y(f) \circ t_c]{} & Y(c') \end{array}$$

Let us pick $u \in \hat{\mathcal{C}}(h_c, X)$, and starts computing.

$$
\begin{aligned}
Y(f) \circ t_c \circ \phi_{c,X}(u) &= Y(f) \circ t_c(u_c(\mathrm{id}_c)) \\
&= Y(f) \circ (t \circ u)_c(\mathrm{id}_c) \\
&= (t \circ u)_{c'}(f) \\
&= (t \circ u)_{c'}(\mathrm{id}_{c'} \circ f) \\
&= (t \circ u \circ f^*)_{c'}(\mathrm{id}_{c'}) \\
&= \phi_{c',Y}(t \circ u \circ f^*) \\
&= \phi_{c',Y} \circ (t \circ - \circ f^*)(u)
\end{aligned}
$$

We encourage the reader to carefully check each step above. This conclude the proof of the Yoneda Lemma. Next, we will move on to some of its consequences, and in particular, we will define the fundamental Yoneda embedding $\mathbf{y} : \mathcal{C} \to \hat{\mathcal{C}}$.

2.2 Consequences

We can define the Yoneda functor

$$
\begin{aligned}
\mathbf{y} : \mathcal{C} &\to \hat{\mathcal{C}} \\
c &\mapsto h_c \\
c \xrightarrow{f} c' &\mapsto f_*
\end{aligned}
$$

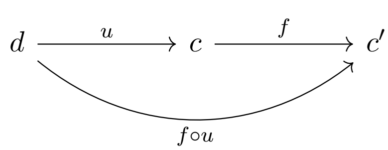
Recall that for $f : c \to c'$, we defined by post-composition, for all $d \in \mathcal{C}$

$$(f^*)_d : h_d(c) \to h_d(c')$$

(we omitted the subscript d, as the natural transformation f^* is the same on every point, but was morally here). This time we have

$$(f_*)_d : h_c(d) \to h_{c'}(d)$$

which is defined by pre-composition as follow.



We can rephrase the Yoneda Lemma using this functor.

Corollary 2.2 (Yoneda Lemma)**.** *There is a natural isomorphism* $\hat{\mathcal{C}}(\mathbf{y}c, X) \simeq X(c)$ *natural in both* $c \in \mathcal{C}$ *and* $X \in \hat{\mathcal{C}}$*.*

Corollary 2.3. *The Yoneda functor* $\mathbf{y}$ *is full and faithful.*

Proof. Let $c, d \in \mathcal{C}$. We apply the Yoneda Lemma for c and $X = \mathbf{y}d$ and we obtain in particular a bijection between $\hat{\mathcal{C}}(\mathbf{y}c, \mathbf{y}d)$ and $\mathbf{y}d(c) = \mathcal{C}(c, d)$, which is precisely what it means to be full and faithful. □

Thus, we have a sort of copy of $\mathcal{C}$ lying inside $\hat{\mathcal{C}}$, that is the image of the Yoneda embedding. The philosophical implications of this embedding is that an object c in a category is the same thing as $\mathbf{y}c$, the data of all morphism with codomain c. There is essentially the same information in an object $c \in \mathcal{C}$ than in the morphisms that point to it. It is equivalent to specify an object of a category or all the morphism that point to it (note that there is a variant with the covariant hom functor h^c, and this time it is about all the morphisms that start from c).

Indeed, this is better understood with the following corollary. It tells us that two things are the same, whenever the the things that point to them are the same.

Corollary 2.4. *Let $c, d \in C$, then $\mathbf{y}c \simeq \mathbf{y}d$ if and only if $c \simeq d$.*

Proof. This is a general fact about fully faithful functor. If $t : \mathbf{y}c \simeq \mathbf{y}d$, then by fullness $t = \mathbf{y}f$ and $t^{-1} = \mathbf{y}g$. We have that $\mathbf{y}(f \circ g) = t \circ t^{-1} = \mathrm{id}$, so by faithfulness, $f \circ g = \mathrm{id}$. Similarly, $g \circ f = \mathrm{id}$. And the converse is always true as functors preserve isomorphisms. □

A presheaf $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ isomorphic to $\mathbf{y}c$ for some $c \in \mathcal{C}$ is called *representable*. Are all presheaves representable? No, but representable presheaves are dense in the following sense.

Theorem 2.5 (Density Formula)**.** *Any presheaf X is a colimit of representable functors.*

Before entering the details of the proof, the intuition is that representable functors are building block of presheaves and we can glue them thanks to colimits. We say that $\mathcal{C}$ is the free co-completion of $\mathcal{C}$, that is co-continuous functor $F : \hat{\mathcal{C}} \to \mathcal{D}$ is uniquely determined by its restriction $F \circ \mathbf{y} : \mathcal{C} \to \mathcal{D}$ via the Yoneda embedding. Let us see that with the infamous examples of graphs. Consider $\mathcal{G}$, a two points $\{e, v\}$ category with two arrows as follow.

$$v \underset{\tau}{\overset{\sigma}{\rightrightarrows}} e$$

A presheaf $G \in \hat{\mathcal{G}}$ is simply

- a set G_V and a set G_E
- a map $G_s : G_E \to G_V$ and a map $G_t : G_E \to G_V$

If we interpret G_V as a set of vertex, G_E as a set of edges, G_s that assigns to each edge its source, and G_t its target, then this is the data of a graph. Conversely, every graph can be described as a presheaf in $\hat{\mathcal{G}}$. Thus $\hat{\mathcal{G}}$ is the category of (directed) graphs. Let us compute the representable graphs $\mathbf{y}v$ and $\mathbf{y}e$.

$\mathbf{y}v_V = \{\mathrm{id}_v\}$ and $\mathbf{y}v_E = \emptyset$, thus $\mathbf{y}v$ is the one point graph with no edge.

$\mathbf{y}e_V = \{\sigma, \tau\}$ and $\mathbf{y}e_E = \{\mathrm{id}_e\}$. $\mathbf{y}e$ has two vertex σ, τ and one edge id_e such that

- $\mathbf{y}e_s(\mathrm{id}_e) = \mathrm{id}_e \circ \sigma = \sigma$
- $\mathbf{y}e_t(\mathrm{id}_e) = \mathrm{id}_e \circ \tau = \tau$

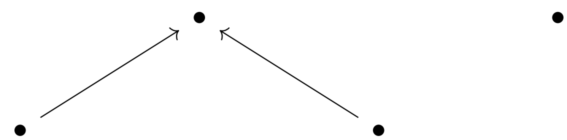
Thus, $\mathbf{y}e$ is the following graph.

$$\sigma \xrightarrow{\mathrm{id}_e} \tau$$

Forgetting about the labels, the two representable graphs are then

$$\bullet \longrightarrow \bullet \qquad\qquad \text{and} \qquad\qquad \bullet$$

The density formula tells us that any graph is the gluing of building block like those. Which is intuitively true, to construct any graph, we add as many $\bullet$ as there are vertex, and then as many $\bullet \to \bullet$ as there are edges, and do the required gluing. For instance, the following graph G

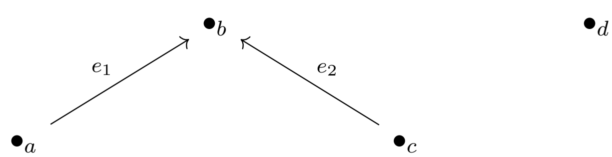


is first constructed by taking two copies of $\mathbf{y}e$ and four of $\mathbf{y}v$, as follow.

Then we identify

- $G_s(e_1) = \bullet_a$
- $G_t(e_1) = \bullet_b = G_t(e_2)$
- $G_s(e_2) = \bullet_c$

to obtain



This process is indeed how we prove the density formula. The Yoneda Lemma indicates that there are $|G(v)| = 4$ natural transformations $\mathbf{y}v$ to G and $|G(e)| = 2$ from $\mathbf{y}e$ to G. For instance we have $a : [\bullet] \to G$ that send the bullet to $\bullet_a$ on G or $e_2 : [\bullet \to \bullet] \to G$ that sends the edge to e_2. This justify the terminology *generalized element* when we refer to a natural transformation of the form $x : \mathbf{y}c \to X$.

For the sake of completeness, we conclude this paper with the proof of the density formula, but it gets a little bit technical and it is no more than what we did above with graphs, but in the general case.

Proof of the density formula. Let $\mathcal{C}$ be a locally small category, and $\hat{\mathcal{C}}$ the category of presheaves. Let us take any presheaf $X : \mathcal{C}^{\text{op}} \to \text{Sets}$. The first step is to unglue all X, that is to take all of its generalized elements $x : \mathbf{y}c \to X$ and put them in one basket. Recall that by Yoneda $x : \mathbf{y}c \to X$ is the same thing as an element of $X(c)$. Thus in our basket, we will put something like $\bigcup\limits_{c \in \mathcal{C}} X(c)$.

More precisely, we introduce the category of elements of X, that we write $\int X$. Its objects are the pairs (c, x) for $c \in \mathcal{C}$ and $x \in X(c)$. A morphism $f : (c, x) \to (d, y)$ is a morphism $f : d \to c$ of $\mathcal{C}$ such that $X(f)(x) = y$. We have an obvious projection

$$\begin{aligned} \pi : \int X &\to \mathcal{C}^{\mathrm{op}} \\ (c, x) &\mapsto c \\ f : (c, x) \to (d, y) &\mapsto f : c \to d \end{aligned}$$

Our claim is that

$$X \simeq \operatorname{colim} \mathbf{y} \circ \pi$$

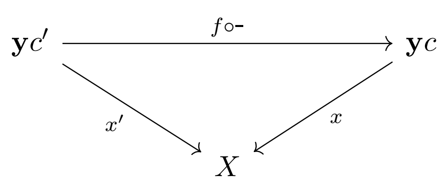
or put otherwise,

$$X \simeq \operatorname*{colim}_{(c,x) \in \int X} \mathbf{y}c$$

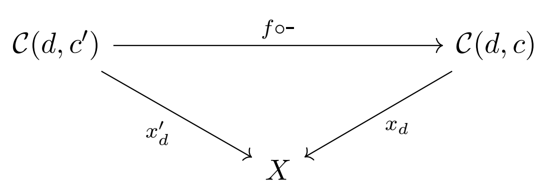
As colimits can be seen as gluing and identifications, this is exactly what happen in the example of our graph. The category of elements had for objects

and we glued them together using the colimit.

Back to the proof. We want to show that X is such a colimit. We first need the injections. The choice is obvious, for $(c, x) \in \int X$, we let $\iota_{(c,x)} = x : \mathbf{y}c \to X$. This data forms indeed a cocone, if $f : (c, x) \to (c', x')$, then



commutes. Indeed, this is a diagram of natural transformation, we will show its commutation by specializing it at any $d \in \mathcal{C}$, and we have



By diagram chasing, we want to prove that for any $u : d \to c'$,

$$x_d(f \circ u) = x'_d(u)$$

But recall that $x_d(f \circ u) = X(f \circ u)(x_c(\mathrm{id}_c))$, hence

$$\begin{aligned} x_d(f \circ u) &= X(u) \circ X(f)(x_c(\mathrm{id}_c)) \\ &= X(u)(x'_{c'}(\mathrm{id}_{c'})) \\ &= x'_d(u) \end{aligned}$$

Here, we often abuse notation and write the same $x : \mathbf{y}c \to X$ and $x_c(\mathrm{id}_c) \in X(c)$ (by Yoneda Lemma). Hence, in the middle equality $X(f)(x_c(\mathrm{id}_c)) = x'_{c'}(\mathrm{id}_{c'})$ is the condition $X(f)(x) = x'$, typed correctly, when x is a natural transformation.

The conditions $X(f)(x) = x'$ are what we did in our graph when required that

- $G_s(e_1) = \bullet_a$
- $G_t(e_1) = \bullet_b = G_t(e_2)$
- $G_s(e_2) = \bullet_c$

We were in fact describing the morphisms in the category $\int G$.

Finally, we conclude by showing this has the universal property of a colimit. Take any presheaf Y with maps $u_{(c,x)} : \mathbf{y}c \to Y$, for $(c, x) \in \int X$. We want to construct a unique natural transformation $t : X \to Y$. We will first define it component-wise, and then show the naturality. Take any $d \in C$. By naturality of the Yoneda Lemma, to define $t_d : X(d) \to Y(d)$ is the same as to define $t_d : \hat{\mathcal{C}}(\mathbf{y}d, X) \to \hat{\mathcal{C}}(\mathbf{y}d, Y)$, and such a map has to send $x : \mathbf{y}d \to X$ to $u_{(d,x)}$. Indeed, we want the following to be commutative.

$$\begin{array}{ccc} \mathbf{y}c & \xrightarrow{x} & X \\ & \searrow^{u_{(c,x)}} & \downarrow t \\ & & Y \end{array}$$

It is the case by definition of t, or to make things more precise, as we are looking at natural transformation from $\mathbf{y}c$ to X, it suffices to check that $t_c(x_c(\mathrm{id}_c)) = u_{(c,x)}(id_c)$, which is precisely how we defined t. Moreover, as each t_c is defined point-wise, everything is unique.

The last thing to check is the naturality. Take any $f : d' \to d$ in $\mathcal{C}$. We want to show that

$$\begin{array}{ccc} \hat{\mathcal{C}}(\mathbf{y}d, X) & \xrightarrow{- \circ f_*} & \hat{\mathcal{C}}(\mathbf{y}d', X) \\ \downarrow t_d & & \downarrow t_{d'} \\ \hat{\mathcal{C}}(\mathbf{y}d, Y) & \xrightarrow[- \circ f_*]{} & \hat{\mathcal{C}}(\mathbf{y}d', Y) \end{array}$$

commutes. Take $x : \mathbf{y}d \to X$, we need to prove that $u_{(d,x)} \circ f_* = u_{(d',x \circ f_*)}$, or that

$$\begin{array}{ccc} \mathbf{y}d' & \xrightarrow{f_*} & \mathbf{y}d \\ & {\scriptstyle u_{(d',x\circ f_*)}} \searrow \quad \swarrow {\scriptstyle u_{(d,x)}} & \\ & Y & \end{array}$$

commutes. This is precisely the cocone commutation condition for Y, providing that $f : d' \to d$ induces indeed a morphism $f : (d, x) \to (d', x \circ f)$, which is the case as $X(f)(x) = x \circ f_*$. Indeed

$$\begin{aligned} X(f)(x_d(\mathrm{id}_d)) &= x_{d'}(f) \\ &= x_{d'}(f \circ \mathrm{id}_{d'}) \\ &= x_{d'} \circ f_*(\mathrm{id}_{d'}) \\ &= (x \circ f_*)_{d'}(\mathrm{id}_{d'}) \end{aligned}$$

□