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Loops and definitions

Definition

Definition (Loop)

A loop Q is a magma where:

- $\exists 1 \in Q \forall a \in Q, a = 1 \cdot a = a \cdot 1$
- $\forall a, b \in A, \exists! x, y \in Q, a \cdot x = b \wedge y \cdot a = b$

Alternatively, if one considers $\forall q \in Q, L_q : x \mapsto q \cdot x$ and $\forall q \in Q, R_q : x \mapsto x \cdot q$, then

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Definition (Alternative definition)

A loop Q is a magma where:

- $\exists 1 \in Q, L_1$ and R_1 are identity functions.
- $\forall q \in Q, L_q$ and R_q are bijections.

Inner Mapping Group

Let Q be a loop.

Definition (Multiplication and Inner Mapping Group)

- $\text{Mlt}(Q) = \langle L_q, R_q \mid q \in Q \rangle$
- $\text{Inn}(Q) = \{ \phi \in \text{Mlt}(Q) \mid \phi(1) = 1 \}$

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$\text{Inn}(Q)$ is a generalization of $\text{Inn}(G) = \{ x \mapsto g^{-1}xg \mid g \in G \} \trianglelefteq \text{Aut}(G)$.

Nilpotency Class

Let Q be a loop.

Definition (Nucleus and Center)

- $\text{Nuc}(Q) = \{a \in Q \mid \forall x, y \in Q, ax \cdot y = a \cdot xy \wedge xa \cdot y = x \cdot ay \wedge xy \cdot a = x \cdot ya\}$
- $Z(Q) = \text{Nuc}(Q) \cap \{a \in Q \mid x \in Q, a \cdot x = x \cdot a\}$

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Definition (Nilpotency Class)

$\text{Nilp}(Q)$ is the smallest integer n such that $Z_n(Q) = Q$, where:

- $Z_0 = \{1\}$
- $Z_{i+1} = \pi^{-1}(Z(Q/Z_i))$ where $\pi : Q \mapsto Q/Z_i$

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Abelian, nilpotent

If $\text{Nilp}(Q) = 1$ then $Q = Z_1 = \pi^{-1}(Z(Q/\{1\})) = Z(Q)$. That is, Q is an abelian group.

AIM conjecture and Csörgő Type Loops

AIM Conjecture

Let Q be a loop.

AIM Conjecture (weak version)

If $\text{Inn}(Q)$ is abelian then $\text{Nilp}(Q) \leq 3$.

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Group theory equivalent

Let G be a group.

- $\text{Inn}(G) \simeq G/Z(G)$
- So $\text{Nilp}(G) = 1 + \text{Nilp}(\text{Inn}(G))$
- If $\text{Inn}(G)$ is abelian, then $\text{Nilp}(G) = 1 + \text{Nilp}(\text{Inn}(G)) = 1 + 1 = 2$

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Some result (Niemenmaa, 2009) [3]

If $\text{Inn}(Q)$ is nilpotent then $\text{Nilp}(Q)$ is finite.

Csörgő Type Loops

Initial AIM conjecture (false)

If $\text{Inn}(Q)$ is nilpotent then $\text{Nilp}(Q) \leq 2$.

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Csörgő counter example

In 2004, Csörgő constructed a loop C of order 128 such that [1]:

- $\text{Inn}(C)$ is abelian

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- $\text{Inn}(C)$ is abelian
- $\text{Nilp}(C) = 3$

Definition (Csörgő Type Loop)

A loop Q is called a Csörgő type loop if

- $\text{Inn}(C)$ is abelian
- $\text{Nilp}(C) \geq 3$

Open problems

Is there a Csörgő type loop

- of order less than 128?
- of odd order?
- of nilpotency class bigger than 3?

Finding smaller Csörgő type loops: Cocycles, central and abelian extensions

Central extensions

Let A be an abelian group and B be a loop.

Definition (Central extension)

If $\theta : B \times B \mapsto A$, we can construct $A :_{\theta} B, = (A \times B, \cdot)$

- $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2 + \theta(b_1, b_2), b_1 + b_2)$

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If θ is such that $\forall x \in B, \theta(0, x) = \theta(x, 0) = 0$, then $A :_{\theta} B$ is a loop [4].

In this case, we call θ a cocycle.

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Definition (Abelian extension)

A more general definition is abelian extension: If $\theta : B \times B \mapsto A$, $\phi, \psi : B \times B \mapsto \text{Aut}(A)$, and $\Gamma = (\phi, \psi, \theta)$ we can construct $A :_{\Gamma} B, = (A \times B, \cdot)$ where

- $(a_1, b_1) \cdot (a_2, b_2) = (\phi_{b_1, b_2}(a_1) + \psi_{b_1, b_2}(a_2) + \theta(b_1, b_2), b_1 + b_2)$

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If $\theta = 0$ and $\phi(b_1, b_2) = \text{Id}_A$, we have, for a good choice of ψ , a semidirect product between A and B .

Constructing Csörgő type loops of nilpotency class 3

Nilpotency class and iterated central extension [4]

A loop is nilpotent if and only if it is an iterated central extension.

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Because our desired loops are nilpotent of class 3, we can hope (?) to construct them by taking three abelian groups A, B and C , two cocycles θ and σ and creating $C = A :_{\theta} (B :_{\sigma} C)$.

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Smallest Csörgő type loops

Currently, the smallest known Csörgő type loops (constructed in [2]) can be decomposed as follow:

- $A = Z_2, B = Z_2 \times Z_2 \times D_8, C = Z_2$
- Then for some morphism $\phi : C \mapsto \text{Aut}(B)$ and for some cocycle μ , $C = A :_{\mu} (B \times_{\phi} C)$ is a Csörgő type loop of order 128.

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Because a semidirect product is an abelian extension, this decomposition is an iterated abelian extension. How to find an iterated central extension?

Finding smaller Csörgő type loops

What kind of loop?

If a smaller Csörgő type loop exists, it seems reasonable to suppose that it has order 64. It's then all about finding A, B, C abelian groups, θ, σ cocycles such that $C = A :_{\theta} (B :_{\sigma} C)$ has order 64 and $\text{Inn}(C)$ abelian.

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Definition (Derived subgroup)

If G is a group, G' is the smallest normal subgroup of G such that G/G' is abelian.

Equivalently, $G' = \langle [g, h], g, h \in G \rangle$.

$|G'|$ measure the *abelianess* of a group.

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Measuring how "AIM" a loop is

If Q is a loop, then $\frac{|\text{Inn}(Q)'|}{|\text{Inn}(Q)|}$ can be used as a metric of how well the loop satisfies the AIM hypothesis.

Why it doesn't work very well

Big space

Let's fix the abelian groups A, B and C such that $C = A :_{\theta} (B :_{\sigma} C)$ has order 64. The space size of the different θ, σ is 2^n where $n \in \{27, 72, 54, 98, 196, 450\}$, depending on A, B, C .

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Uniform metric

For fixed A, B, C and over thousands of samples of σ, θ , the metric usually takes 2-3 different values. How to differentiate the *good* loops from the *bad* loops?

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Cocycle and loop properties

It is moreover not clear how the properties of the cocycle are related to the properties of the loop and how to control the loop properties via the cocycles.

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- Find better metrics (the list of 54 interesting properties?)





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- Find better metrics (the list of 54 interesting properties?)
- Study the link between the cocycles and the resulting loops
- Find the iterated central extensions that lead to the Csörgő type loops of order 128.

References

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Thanks!