Loops of Csörgő Type and the AIM Conjecture

Master's Thesis

Clémence Chanavat

École Centrale Lille
Technical University of Prague
Univesité de Lille

Under the supervision of:
Thibault Gauthier

September 19, 2021

Table of contents I

Loops and definitions

Motivation

Loops

- Generalization of groups theory, without associativity.
- $x \cdot y \cdot z \cdot t$?
- A. A. Albert, The Loop, a community area in Chicago.

Motivation

Loops

- Generalization of groups theory, without associativity.
- $x \cdot y \cdot z \cdot t$?
- A. A. Albert, The Loop, a community area in Chicago.

Applications

- Non-zero octonions are a Moufang loop
- Velocity vector space in special relativity
- Steiner system, sporadic groups.

Definition

Definition (Loop)

A loop Q is a magma where:

- $\exists 1 \in Q\ \forall a \in Q, \quad a = 1 \cdot a = a \cdot 1$
- $\forall a, b \in A, \ \exists ! x, y \in Q, \quad a \cdot x = b \wedge y \cdot a = b$

Alternatively, if one considers $\forall q \in Q, \quad L_q : x \mapsto q \cdot x$ and $\forall q \in Q, \quad R_q : x \mapsto x \cdot q$, then

Definition

Definition (Loop)

A loop Q is a magma where:

- $\exists 1 \in Q \; \forall a \in Q, \quad a = 1 \cdot a = a \cdot 1$
- $\forall a, b \in A, \; \exists! x, y \in Q, \quad a \cdot x = b \wedge \; y \cdot a = b$

Alternatively, if one considers $\forall q \in Q, \quad L_q : x \mapsto q \cdot x$ and $\forall q \in Q, \quad R_q : x \mapsto x \cdot q$, then

Definition (Alternative definition)

A loop Q is a magma where:

- $\exists 1 \in Q, \quad L_1$ and R_1 are identity functions
- $\forall q \in Q, \quad L_q$ and R_q are bijections

Nilpotency Class

Let Q be a loop.

Definition (Nucleus and Center)

- $\mathrm{Nuc}(Q) = \{a \in Q \,|\, \forall x, y \in Q, \;\; ax \cdot y = a \cdot xy \,\wedge\, xa \cdot y = x \cdot ay \,\wedge\, xy \cdot a = x \cdot ya\}$
- $Z(Q) = \mathrm{Nuc}(Q) \cap \{a \in Q \mid x \in Q, \;\; a \cdot x = x \cdot a\}$

Nilpotency Class

Let Q be a loop.

Definition (Nucleus and Center)

- $\mathrm{Nuc}(Q) = \{a \in Q \,|\, \forall x, y \in Q,\ \ ax \cdot y = a \cdot xy \wedge xa \cdot y = x \cdot ay \wedge xy \cdot a = x \cdot ya\}$
- $Z(Q) = \mathrm{Nuc}(Q) \cap \{a \in Q \mid x \in Q,\ \ a \cdot x = x \cdot a\}$

Definition (Nilpotency Class)

$\mathrm{cl}(Q)$ is the smallest integer n such that $Z_n(Q) = Q$, where:

- $Z_0 = \{1\}$
- $Z_{i+1} = \pi^{-1}(Z(Q/Z_i))$ where $\pi : Q \mapsto Q/Z_i$

Nilpotency Class

Let Q be a loop.

Definition (Nucleus and Center)

- $\mathrm{Nuc}(Q) = \{a \in Q \,|\, \forall x, y \in Q,\;\; ax \cdot y = a \cdot xy \,\wedge\, xa \cdot y = x \cdot ay \,\wedge\, xy \cdot a = x \cdot ya\}$
- $Z(Q) = \mathrm{Nuc}(Q) \cap \{a \in Q \mid x \in Q,\;\; a \cdot x = x \cdot a\}$

Definition (Nilpotency Class)

$\mathrm{cl}(Q)$ is the smallest integer n such that $Z_n(Q) = Q$, where:

- $Z_0 = \{1\}$
- $Z_{i+1} = \pi^{-1}(Z(Q/Z_i))$ where $\pi : Q \mapsto Q/Z_i$

Abelian, nilpotent

If $\mathrm{cl}(Q) = 1$ then $Q = Z_1 = \pi^{-1}(Z(Q/\{1\})) = Z(Q)$. That is, Q is an abelian group.

Inner Mapping Group

Let Q be a loop.

Definition (Multiplication and Inner Mapping Group)

- $\mathrm{Mlt}(Q) = \langle L_q, R_q \mid q \in Q \rangle$
- $\mathrm{Inn}(Q) = \{\phi \in \mathrm{Mlt}(Q) \mid \phi(1) = 1\}$

Inner Mapping Group

Let Q be a loop.

Definition (Multiplication and Inner Mapping Group)

- $\mathrm{Mlt}(Q) = \langle L_q, R_q \mid q \in Q \rangle$
- $\mathrm{Inn}(Q) = \{\phi \in \mathrm{Mlt}(Q) \mid \phi(1) = 1\}$

$\mathrm{Inn}(Q)$ is a generalization of $\mathrm{Inn}(G) = \{x \mapsto g^{-1}xg \mid g \in G\} \trianglelefteq \mathrm{Aut}(G)$.

AIM conjecture and Csörgő Type Loops

AIM Conjecture

Let Q be a loop.

AIM Conjecture (weak version)

If $\mathrm{Inn}(Q)$ is abelian (i.e. Q is AIM) then $\mathrm{cl}(Q) \leq 3$.

AIM Conjecture

Let Q be a loop.

AIM Conjecture (weak version)

If $\mathrm{Inn}(Q)$ is abelian (i.e. Q is AIM) then $\mathrm{cl}(Q) \leq 3$.

Group theory equivalent

Let G be a group.

- $\mathrm{Inn}(G) \simeq G/Z(G)$
- So $\mathrm{cl}(G) = 1 + \mathrm{cl}(\mathrm{Inn}(G))$
- If $\mathrm{Inn}(G)$ is abelian, then $\mathrm{cl}(G) = 1 + \mathrm{cl}(\mathrm{Inn}(G)) = 1 + 1 = 2$

AIM Conjecture

Let Q be a loop.

AIM Conjecture (weak version)

If $\mathrm{Inn}(Q)$ is abelian (i.e. Q is AIM) then $\mathrm{cl}(Q) \leq 3$.

Group theory equivalent

Let G be a group.

- $\mathrm{Inn}(G) \simeq G/Z(G)$
- So $\mathrm{cl}(G) = 1 + \mathrm{cl}(\mathrm{Inn}(G))$
- If $\mathrm{Inn}(G)$ is abelian, then $\mathrm{cl}(G) = 1 + \mathrm{cl}(\mathrm{Inn}(G)) = 1 + 1 = 2$

Some result (Niemenmaa, 2009) [3]

If $\mathrm{Inn}(Q)$ is nilpotent then $\mathrm{cl}(Q)$ is finite.

Csörgő Type Loops

Initial AIM conjecture (false)

If $\mathrm{Inn}(Q)$ is abelian then $\mathrm{cl}(Q) \leq 2$.

Csörgő Type Loops

Initial AIM conjecture (false)

If $\mathrm{Inn}(Q)$ is abelian then $\mathrm{cl}(Q) \leq 2$.

Csörgő counter example

In 2004, Csörgő constructed a loop C of order 128 such that [1]:

- $\mathrm{Inn}(C)$ is abelian

Csörgő Type Loops

Initial AIM conjecture (false)

If $\mathrm{Inn}(Q)$ is abelian then $\mathrm{cl}(Q) \leq 2$.

Csörgő counter example

In 2004, Csörgő constructed a loop C of order 128 such that [1]:

- $\mathrm{Inn}(C)$ is abelian
- $\mathrm{cl}(C) = 3$

Csörgő Type Loops

Initial AIM conjecture (false)

If $\text{Inn}(Q)$ is abelian then $\text{cl}(Q) \leq 2$.

Csörgő counter example

In 2004, Csörgő constructed a loop C of order 128 such that [1]:

- $\text{Inn}(C)$ is abelian
- $\text{cl}(C) = 3$

Definition (Csörgő Type Loop)

A loop Q is called a Csörgő type loop if

- $\text{Inn}(C)$ is abelian
- $\text{cl}(C) \geq 3$

Full AIM Conjecture

Rather than just replacing 2 by 3, Michael Kinyon suggested a more structural version of the conjecture.

AIM Conjecture (Full version)

If $\mathrm{Inn}(Q)$ is abelian (i.e. Q is AIM) then $Q\,/\,\mathrm{Nuc}(Q)$ is a group and $Q\,/\,Z(Q)$ is an abelian group.

Note that when Q is AIM, $\mathrm{Nuc}(Q)$ is a normal subloop.

Full AIM Conjecture

Rather than just replacing 2 by 3, Michael Kinyon suggested a more structural version of the conjecture.

AIM Conjecture (Full version)

If $\mathrm{Inn}(Q)$ is abelian (i.e. Q is AIM) then $Q\,/\,\mathrm{Nuc}(Q)$ is a group and $Q\,/\,Z(Q)$ is an abelian group.

Note that when Q is AIM, $\mathrm{Nuc}(Q)$ is a normal subloop. This conjecture holds in particular class of loops (Moufang, LC, extra, automorphic, etc.):

Construction of the known Csörgő loops

The construction goes as follow:

- Take H a (specific) group of order 64 and nilpotency class 2

Construction of the known Csörgő loops

The construction goes as follow:

- Take H a (specific) group of order 64 and nilpotency class 2
- Compute $\mathbb{Z}_2 \times H$ and slightly modify the product law

Construction of the known Csörgő loops

The construction goes as follow:

- Take H a (specific) group of order 64 and nilpotency class 2
- Compute $\mathbb{Z}_2 \times H$ and slightly modify the product law
- The result is a Csörgő loop of order 128

Construction of the known Csörgő loops

The construction goes as follow:

- Take H a (specific) group of order 64 and nilpotency class 2
- Compute $\mathbb{Z}_2 \times H$ and slightly modify the product law
- The result is a Csörgő loop of order 128

Definition (Derived subgroup)

If G is a group, G' is the smallest normal subgroup of G such that G/G' is abelian. Equivalently, $G' = \langle [g,h], \ \ g,h \in G \rangle$. $|G'|$ measure the *abelianess* of a group.

Construction of the known Csörgő loops

The construction goes as follow:

- Take H a (specific) group of order 64 and nilpotency class 2
- Compute $\mathbb{Z}_2 \times H$ and slightly modify the product law
- The result is a Csörgő loop of order 128

Definition (Derived subgroup)

If G is a group, G' is the smallest normal subgroup of G such that G/G' is abelian. Equivalently, $G' = \langle [g, h], \ \ g, h \in G \rangle$. $|G'|$ measure the *abelianess* of a group.

Definition (Elementary groups)

Let p be a prime number. A group G is an elementary abelian p-group if any element of G has an order of p

Construction of the known Csörgő loops

Choose H of order 64 such that

- $H' = Z(H)$
- $H \,/\, H'$ is an elementary abelian 2-group with basis $\{e_1 H', \ldots, e_d H'\} \cong \mathbb{Z}_2^d$
- H' is also an elementary abelian 2-group with basis $\{[e_i, e_j] \,|\, 1 \leq i, j \leq d\}$

Construction of the known Csörgő loops

Choose H of order 64 such that

- $H' = Z(H)$
- $H \,/\, H'$ is an elementary abelian 2-group with basis $\{e_1 H', \ldots, e_d H'\} \cong \mathbb{Z}_2^d$
- H' is also an elementary abelian 2-group with basis $\{[e_i, e_j] \,|\, 1 \leq i, j \leq d\}$

Then considering $f =$ det the determinant of the $(H \,/\, H')^3$ seen as a $\mathbb{Z}_2$ vector space we construct $\mu : H \times H \mapsto Z_2$.

Construction of the known Csörgő loops

Choose H of order 64 such that

- $H' = Z(H)$
- $H \,/\, H'$ is an elementary abelian 2-group with basis $\{e_1 H', \ldots, e_d H'\} \cong \mathbb{Z}_2^d$
- H' is also an elementary abelian 2-group with basis $\{[e_i, e_j] \mid 1 \leq i, j \leq d\}$

Then considering $f = \det$ the determinant of the $(H \,/\, H')^3$ seen as a $\mathbb{Z}_2$ vector space we construct $\mu : H \times H \mapsto Z_2$. Finally,

Csörgő loop

The magma $Q = (\mathbb{Z}_2 \times H, \star)$ such that for any $(a, h), (a', h') \in Q$,

$$(a, h) \star (a', h') = (a + a' + \mu(h, h'), hh')$$

is a Csörgő loop.

Open problems

Is there a Csörgő type loop

- of order less that 128?
- of odd order?
- of nilpotency class bigger than 3?

Finding smaller Csörgő type loops: cocycles, central and abelian extensions

Central extensions

Let A be an abelian group and B be a loop.

Definition (Central extension)

If $\theta : B \times B \mapsto A$, we can construct $A :_\theta B, = (A \times B, \cdot)$

- $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2 + \theta(b_1, b_2), b_1 \cdot b_2)$

Central extensions

Let A be an abelian group and B be a loop.

Definition (Central extension)

If $\theta : B \times B \mapsto A$, we can construct $A :_\theta B, = (A \times B, \cdot)$

- $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2 + \theta(b_1, b_2), b_1 \cdot b_2)$

If θ is such that $\forall x \in B,\ \ \theta(1, x) = \theta(x, 1) = 0$, then $A :_\theta B$ is a loop [4].

In this case, we call θ a cocycle.

Central extensions

Let A be an abelian group and B be a loop.

Definition (Central extension)

If $\theta : B \times B \mapsto A$, we can construct $A :_\theta B, = (A \times B, \cdot)$

- $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2 + \theta(b_1, b_2), b_1 \cdot b_2)$

If θ is such that $\forall x \in B,\ \ \theta(1, x) = \theta(x, 1) = 0$, then $A :_\theta B$ is a loop [4].

In this case, we call θ a cocycle.

Definition (Abelian extension)

A more general definition is abelian extension: If $\theta : B \times B \mapsto A$, $\phi, \psi : B \times B \mapsto \mathrm{Aut}(A)$, and $\Gamma = (\phi, \psi, \theta)$ we can construct $A :_\Gamma B, = (A \times B, \cdot)$ where

- $(a_1, b_1) \cdot (a_2, b_2) = (\phi_{b_1, b_2}(a_1) + \psi_{b_1, b_2}(a_2) + \theta(b_1, b_2), b_1 \cdot b_2)$

Constructing Csörgő type loops of nilpotency class 3

Nilpotency class and iterated central extension [4]

A loop is nilpotent if and only if it is an iterated central extension.

$$L = A_n :_{\theta_n} (A_{n-1} :_{\theta_{n-1}} (\cdots :_{\theta_1} (A_1 :_{\theta_0} A_0))$$

Constructing Csörgő type loops of nilpotency class 3

Nilpotency class and iterated central extension [4]

A loop is nilpotent if and only if it is an iterated central extension.

$$L = A_n :_{\theta_n} (A_{n-1} :_{\theta_{n-1}} (\cdots :_{\theta_1} (A_1 :_{\theta_0} A_0))$$

Because our desired loops are nilpotent of class 3, we can hope to construct them by taking three abelian groups A, B and C, two cocycles θ and σ and creating $L = A :_\theta (B :_\sigma C)$.

Constructing Csörgő type loops of nilpotency class 3

Nilpotency class and iterated central extension [4]

A loop is nilpotent if and only if it is an iterated central extension.

$$L = A_n :_{\theta_n} (A_{n-1} :_{\theta_{n-1}} (\cdots :_{\theta_1} (A_1 :_{\theta_0} A_0))$$

Because our desired loops are nilpotent of class 3, we can hope to construct them by taking three abelian groups A, B and C, two cocycles θ and σ and creating $L = A :_\theta (B :_\sigma C)$.

Smallest Csörgő type loops

Currently, the smallest known Csörgő type loops (constructed in [2]) have the following decomposition:

$$C = \mathbb{Z}_2 :_\mu (\mathbb{Z}_2^3 :_\sigma \mathbb{Z}_2^3)$$

Finding smaller Csörgő type loops

What kind of loop?

If a smaller Csörgő type loop exists, it seems reasonable to suppose that it has order 64. It's then all about finding A, B, C abelian groups, θ, σ cocycles such that $L = A :_\theta (B :_\sigma C)$ has order 64 and $\mathrm{Inn}(L)$ abelian.

Finding smaller Csörgő type loops

What kind of loop?

If a smaller Csörgő type loop exists, it seems reasonable to suppose that it has order 64. It's then all about finding A, B, C abelian groups, θ, σ cocycles such that $L = A :_\theta (B :_\sigma C)$ has order 64 and $\mathrm{Inn}(L)$ abelian.

Measuring how "AIM" a loop is

If Q is a loop, then $|\mathrm{Inn}(Q)'|$ can be used as a metric of how well the loop satisfies the AIM hypothesis.

How to generate smaller Csörgő loops

Big space

Let's fix the abelian groups A, B and C such that $L = A :_\theta (B :_\sigma C)$ has order 64. The space size of the different θ, σ is 2^n where $n \in \{27, 72, 54, 98, 196, 450\}$, depending on A, B, C.

How to generate smaller Csörgő loops

Big space

Let's fix the abelian groups A, B and C such that $L = A :_\theta (B :_\sigma C)$ has order 64. The space size of the different θ, σ is 2^n where $n \in \{27, 72, 54, 98, 196, 450\}$, depending on A, B, C.

Cocycle and loop properties

Thus we need to determine how cocycles influence the resulting loops. Let's focus on simple extension $A :_\theta B$

Simple extension

Let A, B two abelian groups and $\Theta = \{\theta : B \times B \mapsto A \,|\, \forall x \in B,\;\; \theta_{1,x} = 0 = \theta_{x,1}\}$.

Simple extension

Let A, B two abelian groups and $\Theta = \{\theta : B \times B \mapsto A \,|\, \forall x \in B,\ \theta_{1,x} = 0 = \theta_{x,1}\}$.

Definition

- $\theta_1 \sim \theta_2$ iff $(A :_{\theta_1} B) \cong (A :_{\theta_2} B)$
- It's an equivalence relation

Simple extension

Let A, B two abelian groups and $\Theta = \{\theta : B \times B \mapsto A \,|\, \forall x \in B, \;\; \theta_{1,x} = 0 = \theta_{x,1}\}$.

Definition

- $\theta_1 \sim \theta_2$ iff $(A :_{\theta_1} B) \cong (A :_{\theta_2} B)$
- It's an equivalence relation
- Behavior of $\Theta / \sim$

Simple extension

Let A, B two abelian groups and $\Theta = \{\theta : B \times B \mapsto A \,|\, \forall x \in B,\ \ \theta_{1,x} = 0 = \theta_{x,1}\}$.

Definition

- $\theta_1 \sim \theta_2$ iff $(A :_{\theta_1} B) \cong (A :_{\theta_2} B)$
- It's an equivalence relation
- Behavior of $\Theta/\sim$

Definition

Let $\theta, \sigma \in \Theta$. We define the cocycle $\theta \oplus \sigma : B \times B \mapsto A$ by

$$\forall x, y \in B,\ \ (\theta \oplus \sigma)_{x,y} = \theta_{x,y} + \sigma_{x,y}$$

where the $+$ is the addition law on A.

Simple extension

Let A, B two abelian groups and $\Theta = \{\theta : B \times B \mapsto A \,|\, \forall x \in B,\ \ \theta_{1,x} = 0 = \theta_{x,1}\}$.

Definition

- $\theta_1 \sim \theta_2$ iff $(A :_{\theta_1} B) \cong (A :_{\theta_2} B)$
- It's an equivalence relation
- Behavior of $\Theta/\sim$

Definition

Let $\theta, \sigma \in \Theta$. We define the cocycle $\theta \oplus \sigma : B \times B \mapsto A$ by

$$\forall x, y \in B,\ \ (\theta \oplus \sigma)_{x,y} = \theta_{x,y} + \sigma_{x,y}$$

where the $+$ is the addition law on A.

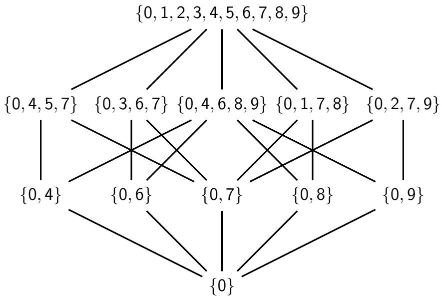
Unfortunately, $(\Theta/\sim, \oplus)$ has no clear structure

Lattice

On the small example $A = B = \mathbb{Z}_3$. $\Theta/\sim = \{T_0, \ldots, T_9\}$.

Lattice

On the small example $A = B = \mathbb{Z}_3$. $\Theta/\sim = \{T_0, \ldots, T_9\}$. We have the following lattice:

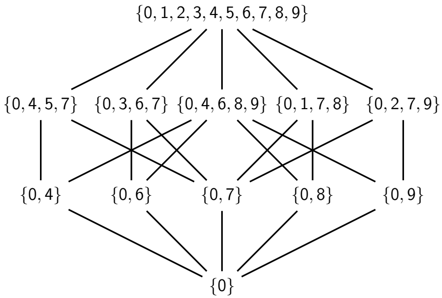


Each point $J = \{j_1, \ldots, j_k\}$ is such that

- $S = (\bigcup_{j \in J} T_j, \oplus)$ is a group

Lattice

On the small example $A = B = \mathbb{Z}_3$. $\Theta/\sim = \{T_0, \ldots, T_9\}$. We have the following lattice:



Each point $J = \{j_1, \ldots, j_k\}$ is such that

- $S = (\bigcup_{j \in J} T_j, \oplus)$ is a group
- $S \cong \mathbb{Z}_3^r$

Lattice structure

Definition

Let $S \subseteq \Theta$. The closure of S is the smallest set $\overline{S}$ containing S that is close under $\oplus$.

Lattice structure

Definition

Let $S \subseteq \Theta$. The closure of S is the smallest set $\overline{S}$ containing S that is close under $\oplus$.

Generalization?

A, B be two abelian groups. $\Theta/\sim = \{T_0, \dots, T_{n-1}\}$ with $T_0 = [(x, y) \mapsto 0]$.

- $(T_0, \oplus)$ is a group.

Lattice structure

Definition

Let $S \subseteq \Theta$. The closure of S is the smallest set $\overline{S}$ containing S that is close under $\oplus$.

Generalization?

A, B be two abelian groups. $\Theta / \sim = \{T_0, \ldots, T_{n-1}\}$ with $T_0 = [(x, y) \mapsto 0]$.

- $(T_0, \oplus)$ is a group.
- There exists $i_0 = i, i_1, \ldots, i_k$ such that $\overline{T_i} = \bigcup_{0 \leq j \leq k} T_{i_j}$

Lattice structure

Definition

- $L_0 = \{\overline{T} \mid T \in \Theta / \sim\}$
- $L_{n+1} = \{\overline{L \cup T} \mid (L, T) \in L_n \times \Theta / \sim\}$

Generalization?

$L = \bigcup_{n \in \mathbb{N}} L_n$ is a modular sublattice of

$$\left\{ \bigcup_{i \in I} T_i \mid I \in \mathcal{P}(\{0, \ldots, n-1\}) \right\}$$

Lattice structure

Definition

- $L_0 = \{\overline{T} \mid T \in \Theta/\sim\}$
- $L_{n+1} = \{\overline{L \cup T} \mid (L, T) \in L_n \times \Theta/\sim\}$

Generalization?

$L = \bigcup_{n \in \mathbb{N}} L_n$ is a modular sublattice of

$$\left\{ \bigcup_{i \in I} T_i \mid I \in \mathcal{P}(\{0, \ldots, n-1\}) \right\}$$

Each element of the lattice L is isomorphic to A^r for some $r > 0$.

Generating Csörgő loops

We describe the decomposition $A :_\theta (B :_\sigma C)$ of 114 groups of order 64, that have nilpotency class 3.

A	B	C	Number of groups
$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	8
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	14
$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	3
$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	69
$\mathbb{Z}_4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	8
$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	3
$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	9

Generating Csörgő loops

We describe the decomposition $A :_\theta (B :_\sigma C)$ of 114 groups of order 64, that have nilpotency class 3.

A	B	C	Number of groups
$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	8
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	14
$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	3
$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	69
$\mathbb{Z}_4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	8
$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	3
$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	9

Fixing A, B, C we set :

$$T_{A,B,C} = \{t(G) \,|\, |G| = 64,\ \ \mathrm{cl}(G) = 3,\ \ G = A :_{t(G)} (B :_{s(G)} C)\}$$

$$S_{A,B,C} = \{s(G) \,|\, |G| = 64,\ \ \mathrm{cl}(G) = 3,\ \ G = A :_{t(G)} (B :_{s(G)} C)\}$$

Generating Csörgő loops

We compute all the extensions:

$$\mathcal{L} = \left\{ A :_\theta (B :_\sigma C) \,|\, (\theta, \sigma) \in \overline{T_{A,B,C}} \times \overline{S_{A,B,C}} \right\}$$

Generating Csörgő loops

We compute all the extensions:

$$\mathcal{L} = \{A :_\theta (B :_\sigma C) \,|\, (\theta, \sigma) \in \overline{T_{A,B,C}} \times \overline{S_{A,B,C}}\}$$

Resulting loops

- Loops close to groups, thanks to the lattice structure.

Generating Csörgő loops

We compute all the extensions:

$$\mathcal{L} = \{A :_\theta (B :_\sigma C) \,|\, (\theta, \sigma) \in \overline{T_{A,B,C}} \times \overline{S_{A,B,C}}\}$$

Resulting loops

- Loops close to groups, thanks to the lattice structure.
- No Csörgő loops were found...

Generating Csörgő loops

We compute all the extensions:

$$\mathcal{L} = \left\{ A :_\theta (B :_\sigma C) \,\middle|\, (\theta, \sigma) \in \overline{T_{A,B,C}} \times \overline{S_{A,B,C}} \right\}$$

Resulting loops

- Loops close to groups, thanks to the lattice structure.
- No Csörgő loops were found...
- But non-associative loops with $|\mathrm{Inn}(Q)'| = 2$

Conclusion

Summary

- Csörgő loops are of interest for the AIM conjecture

Summary

- Csörgő loops are of interest for the AIM conjecture
- We tried to find Csörgő loops of size smaller than 128

Summary

- Csörgő loops are of interest for the AIM conjecture
- We tried to find Csörgő loops of size smaller than 128
- To do that, we exploited algebraic structure of the cocycles

Improvements and contributions

Improvements

- A better understanding of the relationship between loops and cocycles
- Clustering the cocycles via $\sim'$
- A better measure of "AIM closeness"
- Something that doesn't exist?

Improvements and contributions

Improvements

- A better understanding of the relationship between loops and cocycles
- Clustering the cocycles via $\sim'$
- A better measure of "AIM closeness"
- Something that doesn't exist?

Contributions

- Method to easily of lot of loops of interest
- "Almost" Csörgő loops
- Computational tools: GAP and Python

References

- Piroska Csörgő. "Abelian inner mappings and nilpotency class greater than two". In: *European Journal of Combinatorics* 28.3 (2007), pp. 858–867. issn: 0195-6698. doi: https://doi.org/10.1016/j.ejc.2005.12.002. url: https://www.sciencedirect.com/science/article/pii/S0195669805001708.
- Aleš Drápal and Petr Vojtěchovský. *Small loops of nilpotency class three with commutative inner mapping groups*. 2015. arXiv: 1509.05723 [math.GR].
- MARKKU NIEMENMAA. "FINITE LOOPS WITH NILPOTENT INNER MAPPING GROUPS ARE CENTRALLY NILPOTENT". In: *Bulletin of the Australian Mathematical Society* 79.1 (2009), pp. 109–114. doi: 10.1017/S0004972708001093.
- David Stanovský and Petr Vojtěchovský. *Abelian extensions and solvable loops*. 2015. arXiv: 1509.05733 [math.GR].

Thanks!