Diagrammatic (∞, n)-categories

Clémence Chanavat[1]

Tallinn University of Technology

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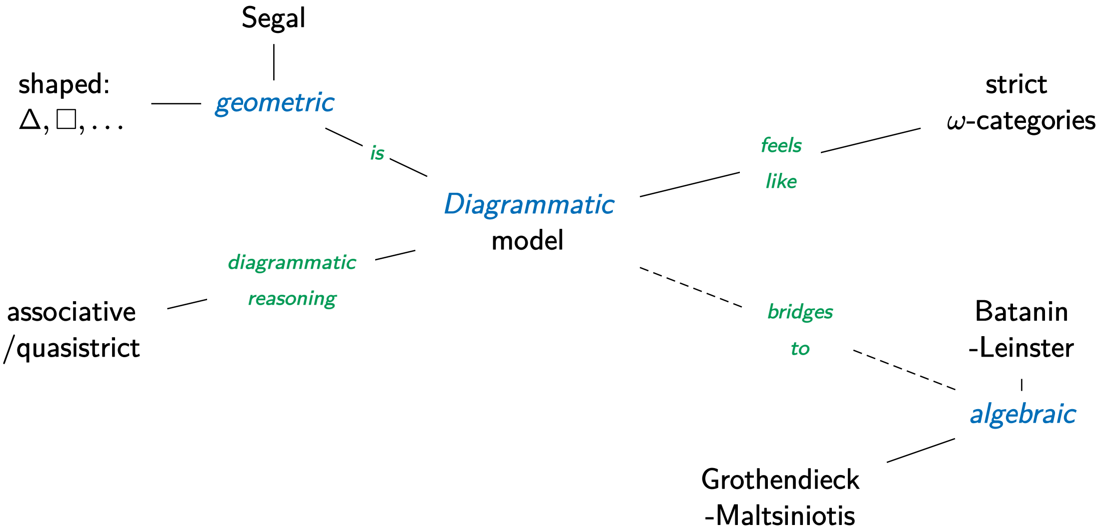
[1] j.w.w. Amar Hadzihasanovic

Landscape of models



Towards a self-contained model

- *No extra data* beyond cells, faces & degeneracies maps;
- Duals, suspensions, Gray products, joins, *all defined representably*;
- Diagrammatic arguments *just work* (or can be tweaked);
- *Explicit* cellular model of the walking equivalence;

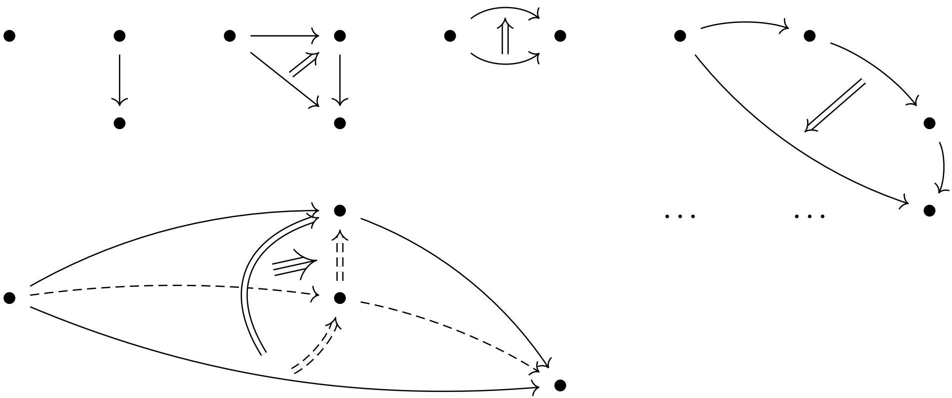
The diagrammatic model

Diagrammatic (∞, n)-categories are based on *diagrammatic sets*.

Definition 1: A *diagrammatic set* X is a presheaf over $\odot$.

The base category ⊙ [Had24]

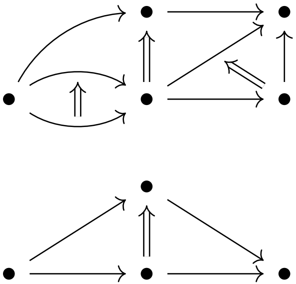
An inductively generated family of *regular directed cell complexes* with a maximal element:



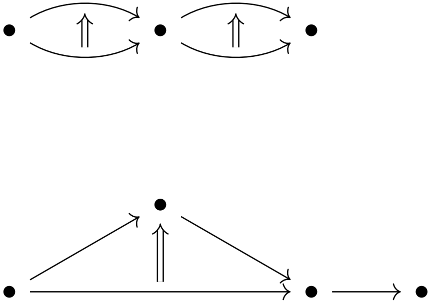
. . . which are stable under *duals*, *Gray products*, *joins*, *suspensions*.

Paste atoms together. . .

. . . and obtain *molecules*:

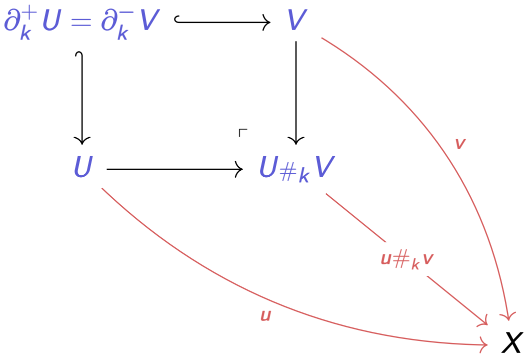


round



not
round

$\odot \hookrightarrow \text{molecules} \hookrightarrow \odot\mathbf{Set}$ $\qquad X \in \odot\mathbf{Set}$



$\partial_k^+ U, U, V, U \#_k V \rightsquigarrow$ molecules

$u, v, u \#_k v \rightsquigarrow$ pasting diagrams in X

Definition 2: A pasting diagram $u\colon U \to X$ is *round* if U is a round molecule.

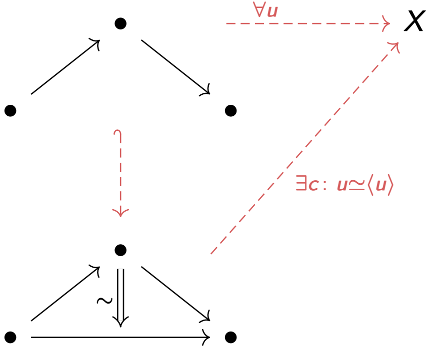
Equivalences [CH24a]

Definition 3: An n-round diagram is an *equivalence* if it is invertible up to $(n+1)$-round diagrams, which are also equivalences.
If there exists an equivalence $e\colon x \Rightarrow y$, we write $x \simeq y$.

Theorem 1:

1. *The relation $\simeq$ is*
   a) *an equivalence relation;*
   b) *a congruence with respect to $- \#_{n-1} -$;*
2. *All degenerate diagrams are equivalences.*

Definition 4: A *diagrammatic (∞,∞)-category* is a diagrammatic set X such that every *round* pasting diagram $u\colon U \to X$ is equivalent to some parallel cell $\langle u \rangle$, called a *weak composite*.



1. The model is *natively (∞,∞)-categorical*.
2. If all cells of dimension $> n$ are equivalences $\rightsquigarrow$ *diagrammatic (∞, n)-categories.*

Definition 5: A *functor* of (∞, n)-categories is a natural transformation of the underlying diagrammatic sets.

Definition 6: A functor $f\colon X \to Y$ is an *ω-equivalence* if

- f is *essentially surjective* on 0-cells;
- for all parallel $u, v \in X$, the induced function

  $$f_{u,v}\colon X(u, v) \to Y(fu, fv)$$

  is *essentially surjective*.

Main results of [CH24b]

Theorem 2 (C., Hadzihasanovic): *For any $n \in \mathbb{N} \cup \{\infty\}$, there exists a model structure on diagrammatic sets where:*

1. *fibrant objects are exactly the (∞, n)-categories;*
2. *weak equivalences between (∞, n)-categories are exactly the ω-equivalences.*

Theorem 3 (C., Hadzihasanovic): *The model structure for $(\infty, 0)$-categories is Quillen equivalent to the classical model structure on simplicial sets.*

Further work

1. Comparison with other models for $n > 0$;
2. Semistrictification via fibrant replacement;
3. Underlying diagrammatic $(\infty, 1)$-category of a model category.

References I

- C. Chanavat and A. Hadzihasanovic, *Equivalences in diagrammatic sets*, September 2024, arxiv:2410.00123.
- ———, *Model structures for diagrammatic (∞, n)-categories*, October 2024, arxiv:2410.19053.
- ———, *Diagrammatic sets as a model of homotopy types*, Homology, Homotopy and Applications (to appear).
- A. Hadzihasanovic, *Combinatorics of higher-categorical diagrams*, April 2024, arXiv:2404.07273.

Thanks!