What is a diagrammatic (∞, ∞)-category?

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Idea

$$\frac{\text{Diagrammatic sets} \quad \textit{are to} \quad (\infty, \infty)\text{-categories}}{\textit{what} \quad \text{reflexive graphs} \quad \textit{are to} \quad \text{categories.}}$$

Corollary

$$\frac{\textit{To go from} \quad \text{reflexive graphs} \quad \textit{to} \quad \text{categories} \quad \textit{is}}{\textit{to go from} \quad \text{diagrammatic sets} \quad \textit{to} \quad (\infty, \infty)\text{-categories}\ldots}$$

. . . if we replace all equalities by *some notion of equivalence*.

Contents I

From reflexive graphs to categories.

Recall

A *reflexive graph* is a graph (edges and vertices), together with, for each vertex v, a designated unit-edge $1_v : v \to v$.

Facts

Definition 1.1: A *category* is a reflexive graph together with a *composition operation* which is *associative* and *unital*.

1. A *composition operation* transforms any *pasting* of 1-cells into a single *1-cell*.
   1.1 *1-cell* is a nickname for *edge*;
   1.2 A *pasting of 1-cells* is a list of 1-cells with compatible *boundaries*;
      1.2.1 The *boundaries* of a 1-cell are the nickname for its *source* and *target*.
2. A composition operation is *unital* if it appropriately respects the *unit*;
3. A composition operation is *associative* if. . .

The construction

To go from reflexive graphs to categories, we will

$$\begin{array}{ccccccc}\text{Reflexive graph} & \rightarrow & \text{boundaries} & \rightarrow & \text{pasting} & \rightarrow & \text{composition operation.} \\ & \rightarrow & \text{units} & & & & \\ & \rightarrow & \text{associativity.} & & & & \end{array}$$

Reflexive graphs

Definition 1.2: The *base category of reflexive graphs*, written $\odot_1$, has 2 objects and 3 non-identity arrows

$$\mathbf{1} \underset{\partial^-}{\overset{\partial^+}{\hookrightarrow}} \twoheadleftarrow \sigma - \vec{I}$$

such that $\sigma \circ \partial^\alpha = \mathrm{id}_{\mathbf{1}}$, for both $\alpha = \pm$.

Definition 1.3: A *reflexive graph* is a presheaf on $\odot_1$. With natural transformations as maps, this forms the category $\odot_1\mathbf{Set}$.

Picturing a reflexive graph

Let $G \colon \odot_1^{\mathrm{op}} \to \mathbf{Set}$ be a reflexive graph. We call

	$G(\mathbf{1})$	the set of *0-cells* of	G
and	$G(\vec{I})$	the set of *1-cells* of	G

Yoneda:

a 0-cell	$x \in G(\mathbf{1})$	is the same as a map	$x : \mathbf{1} \to G$
a 1-cell	$f \in G(\vec{I})$	is the same as a map	$f : \vec{I} \to G.$

If $\quad x \colon \mathbf{1} \to G \quad$ we draw $\quad x$

If $\quad f \colon \vec{I} \to G \quad$ we draw $\quad x \xrightarrow{f} y \quad$ where $\begin{cases} x = G(\partial^-)f, \\ y = G(\partial^+)f. \end{cases}$

Boundaries of 1-cells

Definition 1.4: Let $G\colon \odot_1^{\mathrm{op}} \to \mathbf{Set}$ be a reflexive graph, and $f\colon \vec{I} \to X$ be a 1-cell. The *input 0-boundary* of f, written $\partial_0^- f$, is the composite

$$\mathbf{1} \xhookrightarrow{\partial^-} \vec{I} \xrightarrow{f} G.$$

The *output 0-boundary* of f, written $\partial_0^+ f$, is the composite

$$\mathbf{1} \xhookrightarrow{\partial^+} \vec{I} \xrightarrow{f} G.$$

By Yoneda again, can picture f by:

$$\partial^- f \xrightarrow{f} \partial^+ f$$

Pasting of 1-cells

$\mathbf{1}$ *can be pictured as* $\bullet$;
$\vec{I}$ *can be pictured as* $\bullet \to \bullet$.

Define $\vec{I}\#_0\vec{I}$ as the following pushout in $\odot_1\mathbf{Set}$:

$$\begin{array}{ccc} \mathbf{1} & \overset{\partial^-}{\hookrightarrow} & \vec{I} \\ {\scriptstyle \partial^+}\big\downarrow & & \big\downarrow \\ \vec{I} & \longrightarrow & \vec{I}\#_0\vec{I} \end{array}$$

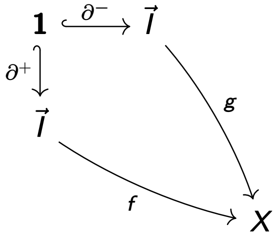
which *can be pictured* as $\bullet \to \bullet \to \bullet$.

Pasting of 1-cells, cont.

The sentence

> Let $f\colon \vec{I} \to G$ and $g\colon \vec{I} \to G$ be two 1-cells such that $\partial_0^+ f = \partial_0^- g$.

translates by saying that the diagram



commutes.
We call $f \#_0 g\colon \vec{I} \#_0 \vec{I} \to G$ the *pasting* of f and g, which can be pictured as $x \xrightarrow{f} y \xrightarrow{g} z$.

Pasting of 1-cells, cont.

More generally, a chain $x \xrightarrow{f_0} y_1 \xrightarrow{f_1} y_2 \cdots\cdots\rightarrow y_n \xrightarrow{f_n} z$ of 1-cells defines a map

$$f_0 \#_0 f_1 \ldots \#_0 f_n \colon \vec{I} \#_0 \vec{I} \#_0 \ldots \#_0 \vec{I} \to X,$$

which we also call a pasting of 1-cells.

Composition operation

Let $G \colon \odot_1^{\mathrm{op}} \to \mathbf{Set}$ be a reflexive graph.

Definition 1.5: A *composition operation* of G is a function $\langle - \rangle$ that maps
any pasting $f_0 \#_0 f_1 \ldots \#_0 f_n$ of 1-cells
to a single 1-cell $\langle f_0 \#_0 f_1 \ldots \#_0 f_n \rangle$.

In picture,

$$
\forall n, \forall (f_i)_{1,\ldots,n} \qquad
\begin{array}{c}
x \xrightarrow{f_0} y_1 \xrightarrow{f_1} y_2 \cdots\cdots y_n \xrightarrow{f_n} z \\
\Big\downarrow \langle - \rangle \\
x \xrightarrow[\langle f_0 \#_0 f_1 \#_0 \cdots \#_0 f_n \rangle]{\hspace{12em}} z
\end{array}
$$

Units

Let $x\colon \mathbf{1} \to G$ be a 0-cell. Then we do

$$\vec{I} \xrightarrow{\sigma} \mathbf{1} \xrightarrow{x} G$$

to obtain $1_x\colon \vec{I} \to G$, the *unit* on x. Picture:

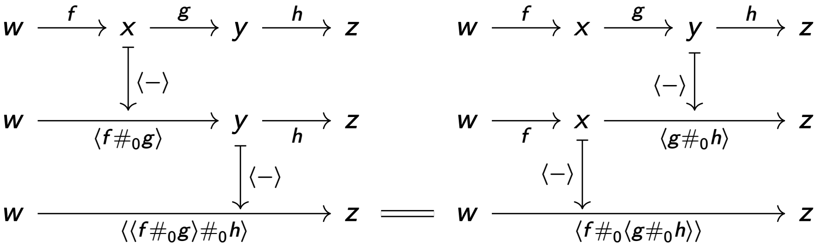
$$\begin{array}{ccc} x & \xrightarrow{1_x} & x \\ & \Downarrow & \\ & x & \end{array} \qquad \begin{array}{c} 1_x := x \circ \sigma\colon \vec{I} \to X \\ \\ x\colon \mathbf{1} \to X \end{array}$$

Definition 1.6: A composition law $\langle - \rangle$ is *unital* if for all 1-cells $f\colon x \to y$,

$$\langle 1_x \#_0 f \rangle = f, \qquad f = \langle f \#_0 1_y \rangle.$$

Associativity

Definition 1.7: A composite law is *associative* if for all $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$,



Summary

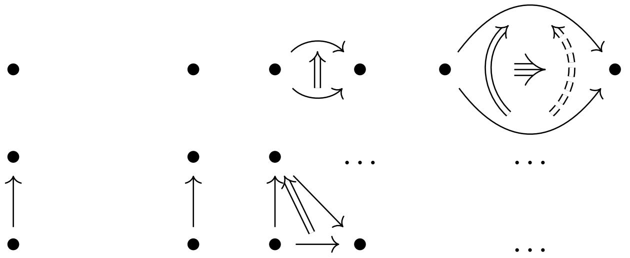
In a *reflexive graph*, there are *boundaries* we can do *pastings* and *take units*.
A *composition law* gives the structure of *category* provided it satisfies two axioms.

The base category of diagrammatic sets

Plan

the base category of reflexive graphs is $\odot_1$.
the base category of diagrammatic sets is *?*

We extend $\odot_1$ to $\odot$ by *induction*.



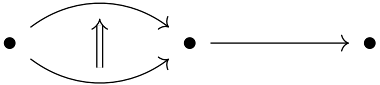
An object of $\odot$ will be called an *atom*.

In fact. . .

By induction we will construct *the molecules*, a subset of which are the atoms.
For the induction to go through, we need that a molecule U comes equipped with:

1. an *input k-boundary* $\partial_k^- U$, (which is also a molecule);
2. an *output k-boundary* $\partial_k^+ U$, (which is also a molecule);
3. a property "*to be round*", which can be true or false.

Example: let U be the following molecule:



What is $\partial_1^- U$? what is $\partial_0^+ U$?
Is U round?

The induction

There are three constructors:

1. *(Point)*;
2. *(Paste)*;
3. *(Rewrite)*;

Point

The point is a molecule.

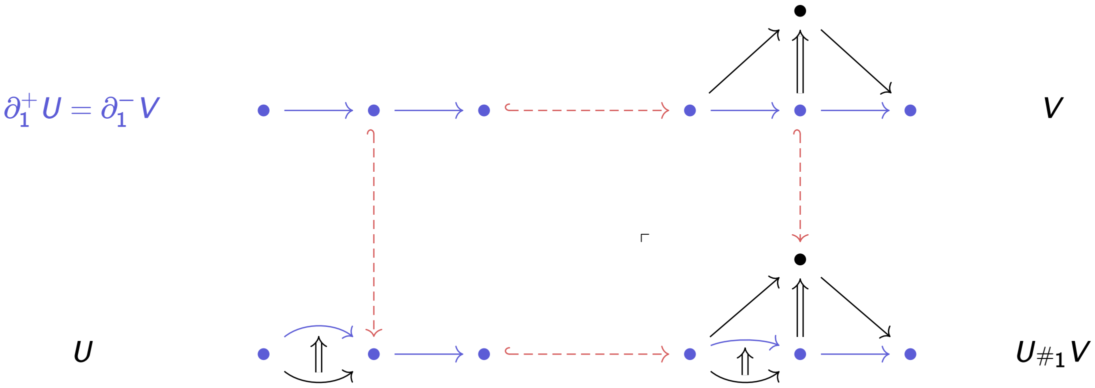
Point: picture

Pasting

Given two molecules U, V and $k \geq 0$ such that $\partial_k^+ U = \partial_k^- V$, define the new molecule $U \#_k V$ as the pushout:

$$\begin{array}{ccc} \partial_k^+ U = \partial_k^- V & \hookrightarrow & V \\ \big\downarrow & & \big\downarrow \\ U & \hookrightarrow & U \#_k V \end{array}$$

Pasting: picture



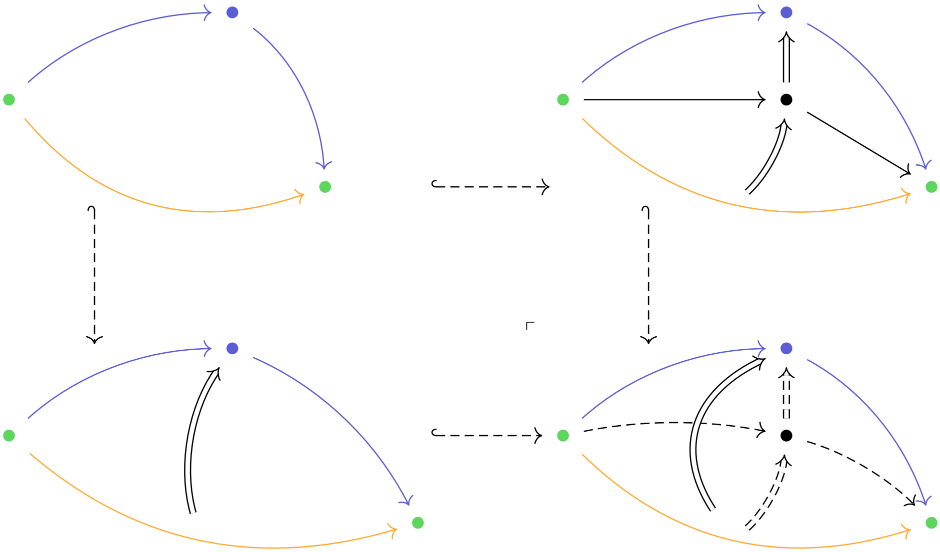
Rewrite

Given two round molecules U, V of the same dimension such that $\partial U = \partial V$, define $\partial(U \Rightarrow V)$ as the pushout:

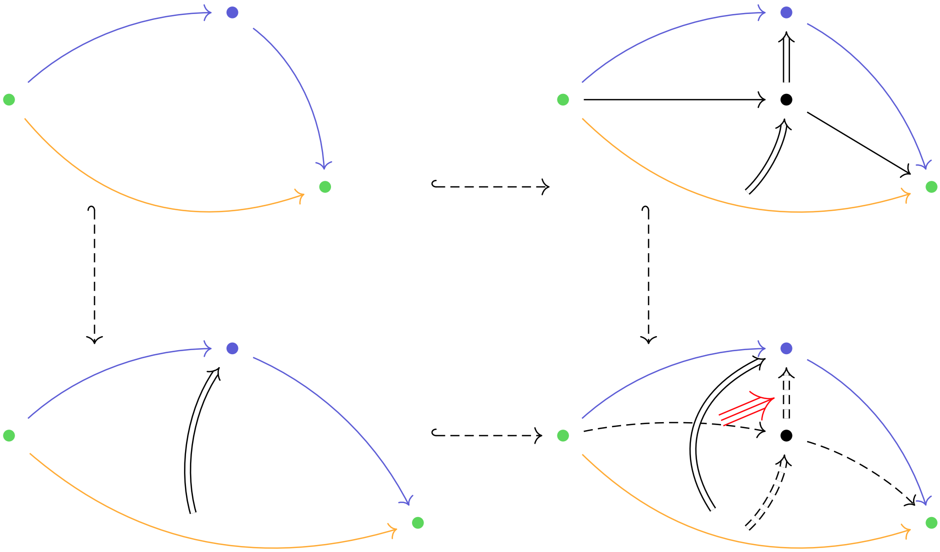
$$\begin{array}{ccc} \partial U = \partial V & \hookrightarrow & V \\ \downarrow & \ulcorner & \downarrow \\ U & \hookrightarrow & \partial(U \Rightarrow V) \end{array}$$

Then define the new molecule $U \Rightarrow V$ by adding a maximal element to $\partial(U \Rightarrow V)$ (plus the correct orientation).

Rewrite: picture



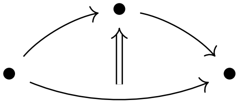
Rewrite: adding the top element



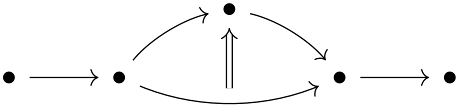
Little summary

Definition 2.1: A *atom* is a molecule with a top-most element, i.e. it is either:

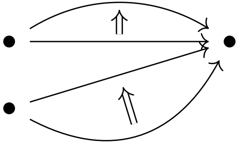
1. the (Point);
2. a (Rewrite) of U and V, written $U \Rightarrow V$.



atom



molecule



regular directed complex

Two kinds of maps: the inclusions. . .

Inclusions take “sub molecules”:

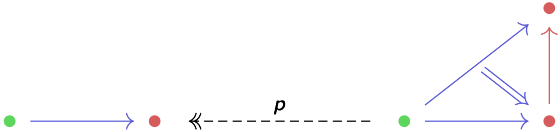
Remark 2.1:

elements $x \in U$ $\overset{\textit{bijection}}{\leftrightsquigarrow}$ inclusions $\iota \colon A \hookrightarrow U$, A atom

using $x \mapsto \mathrm{cl}(x)$.

Two kinds of maps: . . . and the collapses

Collapses “open” part of a molecule:



Eilenberg–Zilber category

We call $\odot$ (a skeleton) of the category of atoms and those maps.

Theorem 2.1 (C., Hadzihasanovic): *The category $\odot$ with those maps is an Eilenberg–Zilber category.*

In particular:

1. unique collapse-inclusion factorisation,
2. no non-identity automorphism,
3. *each collapse has a section, and is entirely determined by them!*

Diagrammatic sets

Definition 2.2: A *diagrammatic set* is a presheaf on $\odot$. A *map* of diagrammatic sets is a natural transformation.

Proposition 2.1: *The Yoneda embedding* $\odot \hookrightarrow \odot\mathbf{Set}$ *factors through:*

$$\odot \hookrightarrow \mathit{molecules} \hookrightarrow \mathbf{RDCpx} \hookrightarrow \odot\mathbf{Set}$$

What is an (∞, ∞)-category?

The construction

To go from reflexive graphs *to* categories, we did

Reflexive graphs	$\to$	boundaries	$\to$	pasting	$\to$	composition operation.
	$\to$	units.				
	$\to$	associativity.				

So *to go from* diagrammatic sets *to* (∞, ∞)-categories, we will:

Diagrammatic sets	$\to$	boundaries	$\to$	pasting	$\to$	composition operation.
	$\to$	units.				
	$\to$	associativity.				

And in the middle, we will introduce the *equivalences*, to replace the *equalities*.

Passage to diagrammatic sets

Let $X : \odot^{\mathrm{op}} \to \mathbf{Set}$ be a diagrammatic set. We call

$$\begin{array}{lll} X(\mathbf{1}) & \text{the set of } \textit{0-cells} \text{ of} & X \\ X(\vec{I}) & \text{the set of } \textit{1-cells} \text{ of} & X \\ \dots & \dots & \\ \bigcup_{\substack{U \text{ atom} \\ \dim U = n}} X(U) & \text{the set of } \textit{n-cells} \text{ of} & X \\ \dots & \dots & \end{array}$$

Yoneda:

$$\begin{array}{llll} \text{a 0-cell} & x \in X(\mathbf{1}) & \text{is the same as a map} & x : \mathbf{1} \to X \\ \text{a 1-cell} & f \in X(\vec{I}) & \text{is the same as a map} & f : \vec{I} \to X \\ & \dots & \dots & \\ \text{a } n\text{-cell} & u \in X(U) & \text{is the same as a map} & u : U \to X \\ & \dots & \dots & \end{array}$$

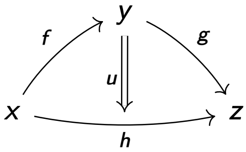
Boudaries

If $u\colon U \to X$ is a diagram, the composite

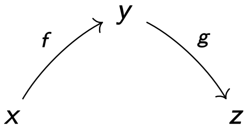
$$\partial_k^\alpha U \hookrightarrow U \xrightarrow{u} X$$

defines $\partial_k^\alpha u$, the *input/output k-boundary* of u. Picture

If $u\colon U \to X$ is



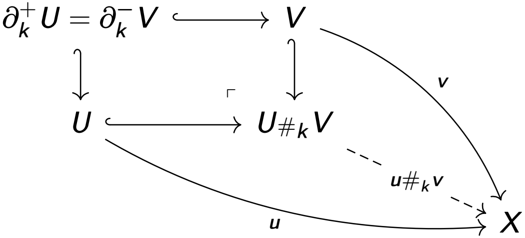
then $\partial_1^- u\colon \partial_1^- U \to X$ is



We write $u\colon v^- \Rightarrow v^+$ *to mean* $\partial^\alpha u = v^\alpha$.

Pasting

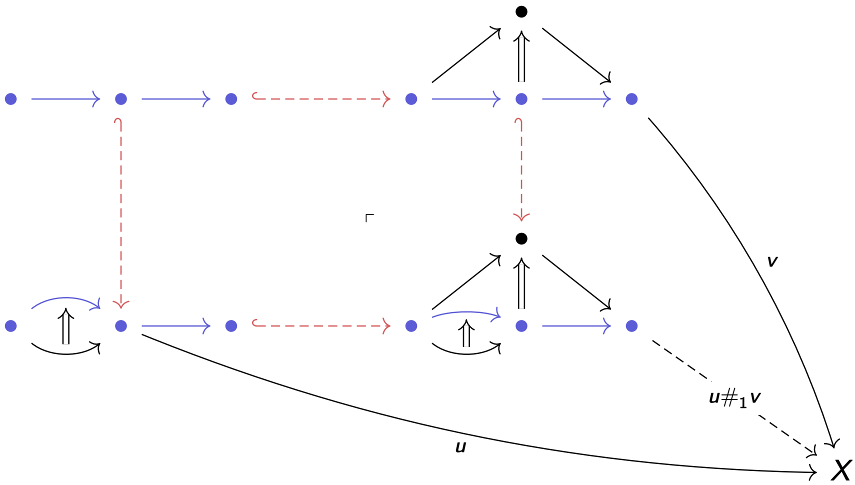
We can similarly paste diagrams together. If $u\colon U \to X$ and $v\colon V \to X$ such that $\partial_k^+ u = \partial_k^- v$, then



Definition 3.1:

If U is an n-dim	*atom*	a map $u\colon U \to X$ is an	n-cell
If U is an n-dim	*round molecule*	a map $u\colon U \to X$ is an	*n-round diagram*
If U is an n-dim	*molecule*	a map $u\colon U \to X$ is an	*n-diagram.*

Pasting: picture



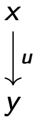
Unit(or)s

For any diagram $u\colon U \to X$, and a collapse $p\colon V \twoheadrightarrow U$, we can take a unit on u by:

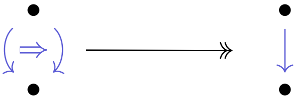
$$V \xrightarrow{p} \!\!\twoheadrightarrow U \xrightarrow{u} X$$

Example:

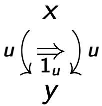
If $u\colon \vec{I} \to X$ is



and $p\colon U \twoheadrightarrow \vec{I}$ is

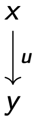


then $u \circ p\colon U \to X$ is

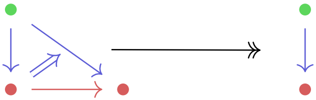


Unitors: other example

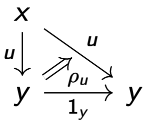
If $u\colon \vec{I} \to X$ is



and $p\colon U \twoheadrightarrow \vec{I}$ is



then $u \circ p\colon U \to X$ is

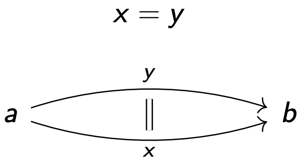


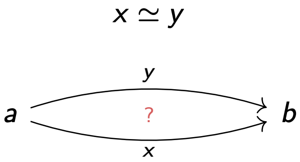
What is an equivalence?

We want to compare two n-cells $x, y \colon U \to X$, but asking "is $x = y$?" is too strong.
Idea: replace $=$ *by* $\simeq$, a looser equivalence relation.

If	$x = y$,	*it is necessary that*	$\partial x = \partial y$	i.e. x and y	are *parallel*
For	$x \simeq y$,	*a prerequisite is*	$\partial x = \partial y$	i.e. x and y	have to be parallel

In picture, we want to fill:



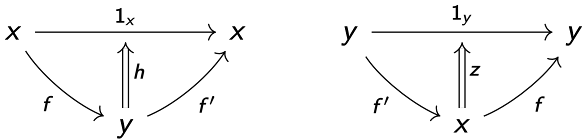


Definition 3.2: Let $x, y \colon U \to X$ be two parallel n-cells. We say that $x \simeq y$ if there exists an $(n+1)$-cell $h \colon x \Rightarrow y$, which is an *equivalence*.

What is an equivalence, in a bicategory?

Definition 3.3: Let $\mathscr{C}$ be a bicategory. We say that a 1-cell $f: x \to y$ is an *equivalence* if:

1. there exists a 1-cell $f': y \to x$, and two 2-cells $h: f *_0 f' \Rightarrow 1_x$ and $z: f' *_0 f \Rightarrow 1_y$:



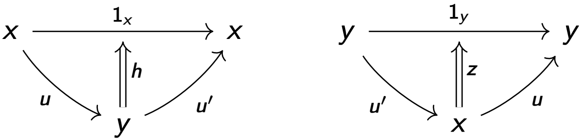
2. h and z are *isomorphisms*.

Notice: an *isomorphism* is nothing more but an an equivalence witnessed by an *equality*.

Equivalences in diagrammatic sets

Definition 3.4: Let X be a diagrammatic set. We say that a n-cell $u\colon x \Rightarrow y$ is an *equivalence* if:

1. there exists a n-cell $u'\colon y \Rightarrow x$, and two $(n+1)$-cells $h\colon u \#_{n-1} u' \Rightarrow 1_x$ and $z\colon u' \#_{n-1} u \Rightarrow 1_y$:



2. h and z are *equivalences*.

Little caveat: for this definition to work well, one should replace "cell" by "round diagram". In an (∞, ∞)-category, this makes no difference.

More on equivalences

Recall: $x \simeq y$ means that there exists an equivalence from x to y.

Theorem 3.1 (C., Hadzihasanovic):

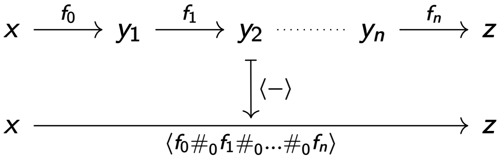
- *The relation $\simeq$ is an equivalence relation;*
- *The pasting of equivalences is again an equivalence;*
- *If $x \simeq y$, and x is an equivalence, then y is an equivalence.*

See [CH24b].

A new flavor of composition:

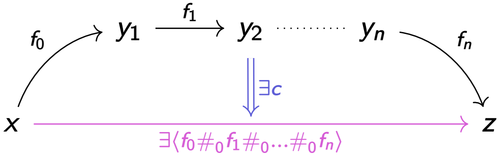
Composition in reflexive graphs:

$\forall n, \forall (f_i)_{1,\ldots,n}$



Composition in diagrammatic sets:

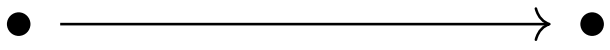
$\forall n, \forall (f_i)_{1,\ldots,n}$

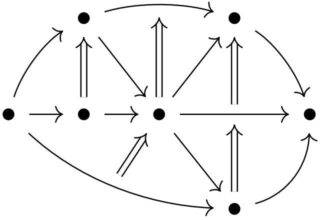
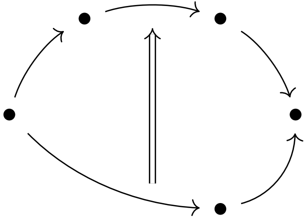


where c is an *equivalence*.

Higher composition shape

By generalisation, for all *round* molecules U, we define the atom $\langle U \rangle := \partial^- U \Rightarrow \partial^+ U$, in picture:

If U is  then $\langle U \rangle$ is 

If U is  then $\langle U \rangle$ is 

Weak composites

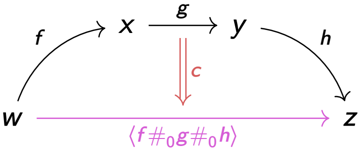
Definition 3.5: Let X be a diagrammatic set, $u\colon U \to X$ be a round diagram.
A *weak composite* for u is:

1. a cell $\langle u \rangle\colon \langle U \rangle \to X$ *parallel to* u,
2. such that $u \simeq \langle u \rangle$.

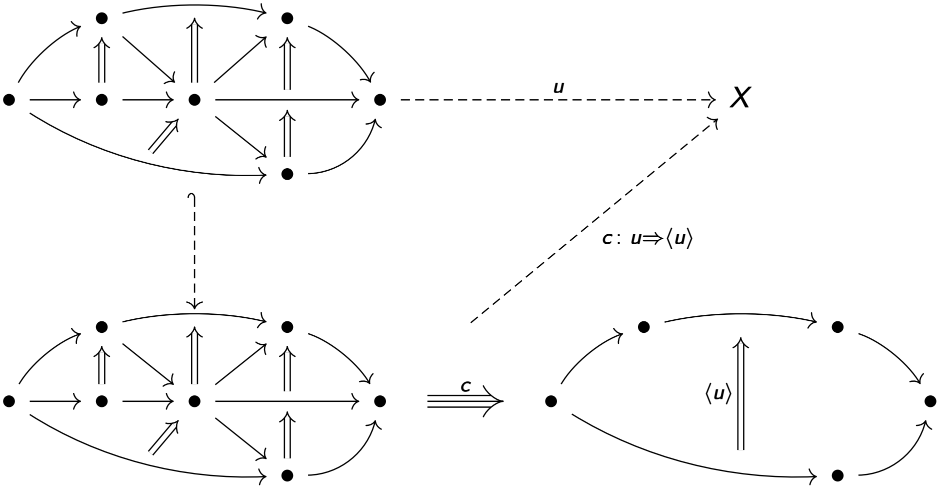
In that case, we say that u *has a weak composite*.

A weak composite for the 1-round diagram $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ is

1. a 1-cell $\langle f \#_0 g \#_0 h \rangle\colon w \Rightarrow z$,
2. an equivalence c fitting in



Other example



(∞, ∞)-category: definition

Definition 3.6 (Hadzihasanovic): A diagrammatic set X is an *(∞, ∞)-category* if

all round diagrams have a weak composite.

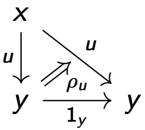
By extension, an *(∞, n)-category* is an (∞, ∞)-category such that all cells of dimension $> n$ are equivalences.

What about unitality, associativity, exchange?

At first glance, an (∞, ∞)-category is “just” a composition operation.

Unitality

Let $u\colon x \to y$ be a 1-cell, we constructed the 2-cell



Theorem 3.2 (C., Hadzihasanovic):

1. *The cell ρ_u is an equivalence, i.e $u \#_0 1_y \simeq u$.*
2. *In fact, all degenerate cells are equivalences.*

Meaning: all diagrammatic sets (hence all (∞, ∞)-categories) are already "unital".
Each surjection $p\colon U \twoheadrightarrow V$ gives a "unit law" for diagrammatic sets.

Composition is unital

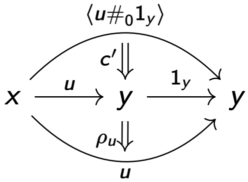
Let $u\colon x \Rightarrow y$ be an n-cell, and $\langle u \#_{n-1} 1_y \rangle$ be a weak composite witnessed by c.

Proof.

Then $\langle u \#_{n-1} 1_y \rangle \overset{\text{by } c}{\simeq} u \#_{n-1} 1_y \overset{\text{by unitor}}{\simeq} u$. Since $\simeq$ is transitive, $\langle u \#_{n-1} 1_y \rangle \simeq u$. □

Picture proof.

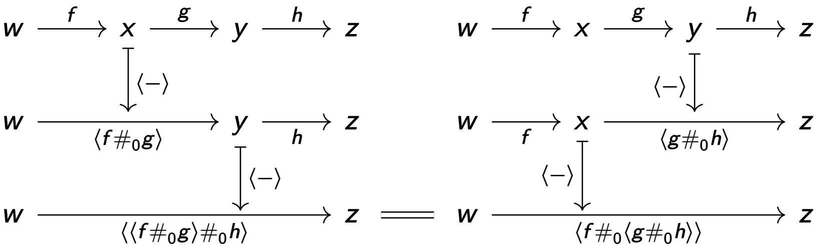
For c' any weak inverse of c, we form the pasting of equivalences:



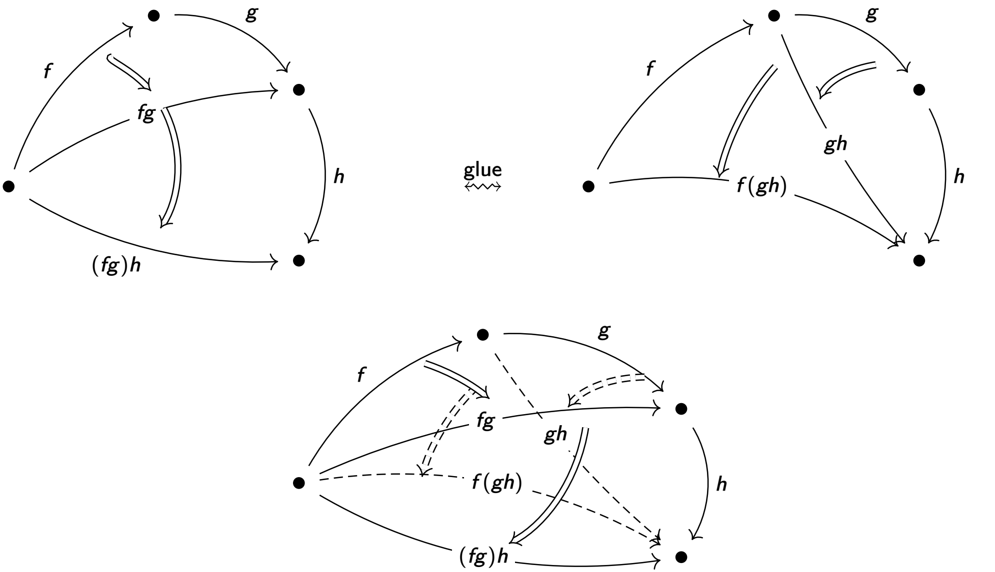
□

Associativity

Consider the pasting of three 1-cells $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ in an (∞, ∞)-category X.
Recall the following picture for reflexive graphs:

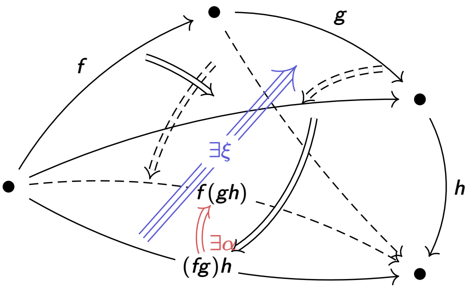


Associativity, cont.



Associativity, cont.

Theorem 3.3: *There exists equivalences α and ξ filling the diagram:*



Theorem 3.4 (C., Hadzihasanovic): *More generally, in a diagrammatic set, every appropriate horn admits such a filler.*

Remark: this is the main result of [CH24b].

Summary

In a *reflexive graph*, there are *boundaries* we can do *pastings* and *take units*.
A *composition law* gives the structure of *category* provided it satisfies two axioms.

In a *diagrammatic set*, there are *boundaries* we can do *pastings* and *take units*.
A *composition law* gives the structure of (∞, ∞)-*category* provided *nothing*.

Theory of (∞, ∞)-categories

Functor

Definition 4.1: A *functor* $f\colon X \to Y$ between two (∞, ∞)-categories is a natural transformation of diagrammatic sets.

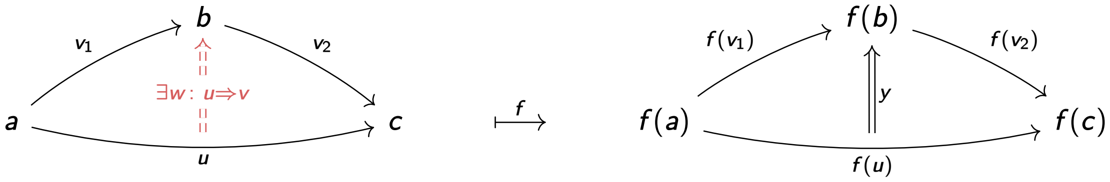
Remark 4.1: Functors preserves:

1. shapes of diagrams, i.e. $(u\colon U \to X) \mapsto (f(x)\colon U \to Y)$;
2. boundaries and unit(or)s, (by naturality);
3. hence equivalences;
4. hence weak composites.

Equivalence of (∞, ∞)-categories

Let $f\colon X \to Y$ be a functor. We say that f is an *ω-equivalence* if:

$\forall (u, v)\colon \partial U \to X$ $\qquad\qquad$ $\forall y\colon f(u) \Rightarrow f(v)$



such that $y \simeq f(w)$.

i.e, f is *essentially surjective* at all dimension.

Notice, we can prove that if $f(x) \simeq f(y)$ then $x \simeq y$, i.e. f is *essentially injective*.

Theorems

We can finally state the main results of [CH24a, CH24c].

Theorem 4.1 (C., Hadzihasanovic): *For any $n \in \mathbb{N} \cup \{\infty\}$, there exists a model structure on diagrammatic sets where:*

1. *fibrant objects are exactly the (∞, n)-categories;*
2. *weak equivalences between (∞, n)-categories are exactly the ω-equivalences.*

Theorem 4.2 (C., Hadzihasanovic): *The model structure for $(\infty, 0)$-categories is Quillen equivalent to the classical model structure on simplicial sets.*

Our theory of $(\infty, 0)$-categories coincides with the classical theory of ∞-groupoids!

Further directions

- Does our notion of (∞, n)-categories coincides with the "classical" notions of (∞, n)-categories?
- Semistrictification: composition *go back to being* an operation.

References I

- C. Chanavat and A. Hadzihasanovic, *Diagrammatic sets as a model of homotopy types*, July 2024, arxiv:2407.06285.
- ———, *Equivalences in diagrammatic sets*, September 2024, arxiv:2410.00123.
- ———, *Model structures for diagrammatic* (∞, n)*-categories*, October 2024, arxiv:2410.19053.

Thanks!