Exercises, week 1.

Throughout this exercise sheet, C and D denote mathematical statements.

Exercise 1: Use truth tables to show that:

1. $\neg(C \wedge D)$ has the same truth value as $\neg C \vee \neg D$.
2. $\neg(C \vee D)$ has the same truth value as $\neg C \wedge \neg D$.
3. $\neg\neg C$ has the same truth value as C.
4. $C \vee \neg C$ is always true.
5. $C \wedge \neg C$ is always false.
6. $C \Rightarrow D$ has the same truth values as $D \vee \neg C$.

Finally, without doing a truth table, but using the above results, prove that:

1. $\neg(C \Rightarrow D)$ is the same as $\neg D \wedge C$.
2. Prove that $C \Rightarrow D$ is the same thing as $\neg D \Rightarrow \neg C$.

Exercise 2: If and only if is two implications in disguise. Prove that $(C \Rightarrow D) \wedge (D \Rightarrow C)$ has the same truth values as $C \iff D$. Is $C \Rightarrow D$ saying the same thing as $D \Rightarrow C$?

The connectives $\wedge, \vee, \Rightarrow$ are the one we use the most to create mathematical proofs and statements. But nothing prevents us to create other ones. We usually do not use them when doing proofs, but they become relevant when learning about *boolean algebra*.

Definition 1: We define a new operator, called the *exclusive or*, or *xor*. Its symbol is $\oplus$. It takes two inputs C, D, and returns $C \oplus D$ which is true as long as *exactly one* of C or D is true. That is, it has the following truth table.

C	D	$C \oplus D$
0	0	0
0	1	1
1	0	1
1	1	0

Please notice the difference with the *or* connective.

Exercise 3: Prove that $C \oplus D$ is the same thing as $(C \vee D) \wedge \neg(C \wedge D)$, which is also the same thing as $\neg(C \iff D)$. It means that this $\oplus$ is in fact not a brand new thing, but that we can reconstruct it from more elementary operations.

Exercise 4: Prove that $C \oplus C$ is always false. Do it using truth tables, then without truth tables, using Exercise 3 and 1

Exercise 5: Here is a more difficult and fun exercise. Define the *nand* connective, written $\bar{\wedge}$, with the following truth table:

C	D	$C \bar{\wedge} D$
0	0	1
0	1	0
1	0	0
1	1	0

1. Check that $C \bar{\wedge} C$ has the same truth values as $\neg C$.
2. Check that $(C \bar{\wedge} D) \bar{\wedge} (C \bar{\wedge} D)$ has the same truth values as $C \wedge D$.
3. Can you construct an expression using only the symbols $C, D, \bar{\wedge}$ that has the same truth values as $C \vee D$.
4. Same question with $C \Rightarrow D$.

We see that in some sense that the $\bar{\wedge}$ connective is universal. Its sole use allows us to construct any truth table we like. Not all connectives enjoy this property. For instance, using only $\wedge$'s, we will never be able to construct a $\vee$. Can you find another such universal connective (hint: there exists only one other)?
As an application, it means for instance that if we were to build a microprocessor, we would not have to bother about crafting individuals $\vee, \wedge, \neg, \ldots$, gates. We would just need nand-gates and we could assemble them together to create the other logical operations.