Exercises, week 2.

Throughout this exercise sheet, $A, B, C, \ldots$ denote mathematical statements.
In this next following exercise, your proof should be one-liners.

Exercise 1: Show that you know how to use the connectives, that is, write the proof of the following, using the methods we present in class.

1. Prove that if A and B, then A.
2. Prove that if A, then $A \vee B$.
3. Prove that if (if A then B) and A, then B.

Now, write proofs of the real following statements. They should follow the same pattern as the one you wrote above, except that you populate them with actual mathematical content.

1. Let $n \in \mathbb{N}$. Prove that if $n^2 = n$ and $n \neq 0$, then $n = 1$.
2. Let $n \in \mathbb{N}$. Prove that if $n^2 = n$ then $n = 1$ or $n = 0$.

Exercise 2: Prove that

$$\forall x \in \{t \in \mathbb{R} \mid t^2 - 2t = 0\}, x + 1 \text{ is odd.}$$

Exercise 3: Prove that there exists a solution to the equation

$$x^2 - \frac{5}{2}x + 1 = 0,$$

which is also a natural number.

Exercise 4:

1. Look at the following mathematical statement:

   If n is even, then n modulo 4 is either 0 or 2.

2. Transform it into a formal statement using quantifier.
3. Prove it.

This next one tries to confuse you on purpose. However, often, real statements will feel as confusing.

Exercise 5: Let $f : \mathbb{N} \to \mathbb{N}$ be a function (something that turns an input element into an output). Prove that for all $n \in \mathbb{N}$ such that n is not zero, either for all natural number p, $f(p) = n$, or that there exists some value q in $\mathbb{N}$ for which $\frac{f(q)}{n}$ is an integer. Hint: try to reduce it to a formal sentence and use the method.

Here is a conceptual exercise. We prove by induction, that we can improve the concept of induction.

Exercise 6: Use induction to prove the (very useful in practice) strong induction principle,

$$(P(0) \wedge (\forall n \in \mathbb{N}, (\forall m \leq n, P(m)) \Rightarrow P(n+1))) \Rightarrow \forall n \in \mathbb{N}, P(n).$$

Exercise 7: Use the strong induction principle to prove that any natural number $n > 1$ decomposes as a product of prime numbers.

The last exercise is experimental, I do not know its pedagogical value. To do math, you do not really *need* to know what concepts are about. Suppose I give you the two following theorems.

Theorem 1: If a group $(G, \cdot, e)$ is such that for all $x \in G$, $x \cdot x = e$, then G is abelian.

Theorem 2: If we let $G := \{0,1\}$, $e := 0$, with the multiplication law defined, for $x, y \in \{0,1\}$,

$$x \cdot y := x \oplus y$$

(the exclusive or), then we obtain a group.

Exercise 8: Combine the two theorems and do two elementary computations to show that the group described in 2 is abelian.

If you need more exercises, ask me.