# Exercises, week 3.

**Exercise 1:** Prove that, for all sets $X, Y, Z$, we have

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z).$$

Prove that for all $A, B \subseteq X$, we have

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B).$$

Does it remind you of something?

We see what happens if we consider $\Omega$ the set of all sets: we can break math.

**Exercise 2:** Let $\Omega$ be the set of all sets. We define by replacement

$$\Omega_0 := \{X \in \Omega \mid X \notin X\}.$$

1. Prove that if $\Omega_0 \in \Omega_0$ then $\Omega_0 \notin \Omega_0$.
2. Prove that if $\Omega_0 \notin \Omega_0$ then $\Omega_0 \in \Omega_0$.
3. Deduce that math is broken.

Note: therefore, we cannot consider the set of all sets, and we have to be careful to use replacement in general. However, such subtelties do generally do not arise in practice.

**Exercise 3:** Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(x) = 2x$. What is the set $f(\mathbb{N})$? Let $\mathcal{O}$ be the subset of $\mathbb{N}$ constituted of odd numbers. What is the set $f^{-1}(\mathcal{O})$?

**Exercise 4:** Let $f : X \to Y$ be a function, let $A, A' \subseteq X$ and $B, B' \subseteq Y$. Prove some of the following identities (they are very useful to know, or at least remember they exist).

- $f(A \cup A') = f(A) \cup f(A)$.
- $f(A \cap A') \subseteq f(A) \cap f(A')$.
- $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$.
- $f^{-1}(B \cap B') \subseteq f^{-1}(B) \cap f^{-1}(B')$.
- $f^{-1}(f(X)) = X$.
- $f^{-1}(f(A)) \supseteq A$.
- $f(f^{-1}(Y)) = f(X)$.
- $f(f^{-1}(B)) \subseteq B$.

It is even a better exercise to try to come up with a example where the full equality fails, for instance provide a function where we do not have $f(A \cap B) = f(A) \cap f(B)$. For a more exhaustive list of these relations, see this Wikipedia page.