Exercises, week 4.

Exercise 1: Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that

1. If f and g are injections then $g \circ f$ is an injection.
2. If f and g are surjections then $g \circ f$ is a surjection.
3. If f and g are bijections then $g \circ f$ is a bijection.

Exercise 2: Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that

1. If $g \circ f$ is injective, then f is injective.
2. If $g \circ f$ is surjective, then g is surjective.
3. If $g \circ f$ is bijective, then f is injective and g is surjective.

Exercise 3: Find a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Exercise 4: Determine if the following functions are injective, surjective, bijective, or none. We call $\mathbb{R}^+$ the set of real numbers greater or equal to 0.

- $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$.
- $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(x) = x^2$.
- $f : \mathbb{R} \to \mathbb{R}^+$ such that $f(x) = x^2$.
- $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) = x^2$.

Exercise 5: Let X be a set.

1. Find an equivalence relation $\sim$ on X such that $X/_\sim$ is
$$\{\{x\} \mid x \in X\}.$$
2. Find an equivalence relation $\sim$ on X such that $X/_\sim$ is
$$\{\{X\}\}.$$

The rest of the exercises are more difficult and conceptual. First, let us secretly do group theory:

Exercise 6: Let X be a set, and call $\mathrm{End}(X)$ the set of all functions from X to itself, that is:

$$\mathrm{End}(X) := \{f : X \to X \mid f \text{ is a function}\}.$$

We define the binary relation $\sim$ on $\mathrm{End}(X)$ by:

$$f \sim g \iff \exists \varphi : X \to X, \varphi \text{ is bijective, and } f = \varphi^{-1} \circ g \circ \varphi.$$

Prove that $\sim$ is an equivalence relation. Hint: if φ and θ are bijections, what is $(\varphi \circ \theta)^{-1}$?

Now we introduce the notion of partition, and we progressively show that in fact, we are talking about equivalence relations in a different way.

Definition 1: Let X be a set. A *partition* of X is a collection of nonempty subsets $\{U_i\}_{i \in I}$ of X, such that $\bigcup_{i \in I} U_i = X$, and if $i \neq j$, $U_i \cap U_j = \emptyset$.

This exercise proves that equivalence classes form partitions.

Exercise 7: Let $(X, \sim)$ be a set with an equivalence relation.

1. Let $x, x' \in X$, show that $[x] = [x']$ or $[x] \cap [x'] = \emptyset$.

2. Understand that the above statement means that if $s, s' \in X/_\sim$, then if $s \neq s'$ we have $s \cap s' = \emptyset$, therefore the set equivalence classes forms a partition of X.

This exercise proves that partitions form equivalence relations.

Exercise 8: Let $\{U_i\}_{i\in I}$ be a partition of a set X. Prove that the binary relation $\sim$ on X defined by

$$x \sim y \iff \exists i \in I, x \in U_i \wedge y \in U_i,$$

is an equivalence relation on X.

This last exercise shows that the two previous exercises are undoing each other, and establishes a theorem.

Exercise 9:

1. Let $(X, \sim)$ be a set with an equivalence relation. Exercise 7 shows that $X/_\sim$ forms a partition. Show that the equivalence relation $\sim'$ we get from this partition according to Exercise 8 is the same as $\sim$.

2. Let $\{U_i\}_{i\in I}$ be a partition of a set X. Exercise 8 shows that we can form an equivalence relation $\sim$ from this partition. Show that the partition given by $\sim$ according to Exercise 7 is the same as $\{U_i\}_{i\in I}$.

Finally, prove the following:

Theorem 2: Let X be a set. There is a bijection between the set of all partitions of X, and the set of all equivalence relations on X.