Exercises, week 9.

Exercise 1: Let G be a group. Let $x, y, z \in G$. Prove that if $x \cdot y = z \cdot y$, then $x = z$.

Definition 1: A group G is *abelian* if for all $x, y \in G$, we have

$$x \cdot y = y \cdot x.$$

Exercise 2: Let G be a group in which for all x, $x \cdot x = e$. Prove that G is abelian.

Definition 2: Let $x \in G$ be an element of a group. Let $n \in \mathbb{Z}$, we define by induction x^n. If $n = 0$, we let $x^0 = e$. Suppose x^n has been defined, we define x^{n+1} to be $x^n \cdot x$. Suppose $n < 0$, then we define $x^n = (x^{-n})^{-1}$.

Exercise 3: Check that this definition satisfies the usual laws of powers, that is

1. $x^n x^m = x^{n+m}$
2. $(x^n)^m = x^{nm}$
3. $x^{-n} = (x^n)^{-1}$

Exercise 4: Let G be a group, let $n, m \in \mathbb{N}$ such that $\gcd(m, n) = 1$. Suppose there exists an $x \in G$ with $x^m = e$. Prove that there exists $y \in G$ such that $y^n = x$.

Exercise 5: Let $f : G \to H$ be a group morphism. Let $x \in G$, show that

$$f(x^{-1}) = f(x)^{-1}.$$

Exercise 6:

1. Let G be a group. Show that $\mathrm{id}_G : G \to G$ is a group morphism.
2. Let $f : G \to H$ and $g : H \to K$ be group morphisms. Show that $g \circ f : G \to H$ is a group morphism.

Exercise 7: Let X be a set. Show that the set of functions:

$$\mathrm{Bij}(X) := \{f : X \to X \mid f \text{ is bijective } \},$$

is a group, whose laws and neutral element you will have to find.