Homework 1.

In this homework, we do fun things: we put mathematics back inside mathematics via Topos theory and Kripke-Joyal semantics (Google both). As we are only familiar with set theory, we only do the work for the (terminal) topos of them all, the topos of sets. You should not attempt this homework if you:

1. Do not care about math.
2. Are only passing this course because it is mandatory.
3. Only care about applications. We are doing the opposite of applications in this homework, we are doing *co*-applications.

You should be able to follow through all the questions purely from what we saw in class. We will introduce a few definitions here and there, but they will only concern sets and functions, so no new concept really. Then, at the end, we will do the same question as in the manageable homework, that is, we will hint towards the truth on why

$$\forall x, (P(x) \wedge Q(x))$$

and

$$(\forall x, P(x)) \wedge (\forall x, Q(x))$$

are the same: *(co)right adjoints preserves (co)limits* (of course, by coright I mean left). If you have fun doing this homework, and want to know why things are the way they are in the universe, then come to the category theory class we are teaching next semester. See also my master's thesis at http://chanavat.site/files/lmfi/lmfi-thesis.pdf. This homework will morally re-do the Section 2.

If you try to do this homework instead of the manageable one, you can have full credits without doing all questions, as long as it is clear that you put as much effort as everyone else in your work. Let's get started.

Throughout this homework, we use the following notations.

Notations 1:

1. We call $\mathbf{1}$ the set with one element, that we call, for the occasion, $\star$. That is:
$$\mathbf{1} := \{\star\}.$$
2. We call Ω the set with two elements, that we call, for the occasion, 0 and 1. That is:
$$\Omega := \{0, 1\}.$$
3. We call $\mathrm{true} : \mathbf{1} \to \Omega$ the function that sends $\star$ to 1. Note: now you keep in a corner of your mind that the 1 will be interpreted as true, and 0 as false. We will see precisely how later.

1 Pullback preserves monomorphisms

We need to define one very important concept before we can go anywhere else.

Definition 2: Let $f : X \to Z$ and $g : Y \to Z$ be two functions. We define the *pullback* of f and g to be the set

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

We call

$$\begin{aligned} p_X : X \times_Z Y &\to X \\ (x, y) &\mapsto x \end{aligned}$$

the *first projection* of the pullback of f and g, and

$$\begin{aligned} p_Y : X \times_Z Y &\to Y \\ (x, y) &\mapsto y \end{aligned}$$

the *second projection* of the pullback.

Exercise 1: Let X be a set.

1. Let $f, g : X \to \mathbf{1}$. Prove that $f = g$.
2. Construct a function from X to $\mathbf{1}$.

Therefore, we proved that there is exactly one function from X to $\mathbf{1}$. This motivates the following definition.

Definition 3: Let X be a set. We call $!_X : X \to \mathbf{1}$ the unique function according to Exercise 1.

We reveal some nice properties of the pullback.

Exercise 2: Let X, Y be two sets.

1. Prove that the pullback of $!_X$ and $!_Y$ is equal to $X \times Y$.
2. Let $f : X \to Z$ and $g : Y \to Z$ be two functions. Prove that
$$f \circ p_X = g \circ p_Y.$$
3. Let $f : X \to Z$ and $g : Y \to Z$ be two functions. Suppose that f is injective. Prove that p_X is injective.

2 The subobject classifier

We will progressively understand why the next definition is called like that.

Definition 4: Let X be a set. We call any function $\phi : X \to \Omega$ a *formula*. If $\phi : X \to \Omega$ is a formula, we call the subset of X

$$X_\phi := \{x \in X \mid \phi(x) = 1\}.$$

the *domain of validity* of ϕ.

Exercise 3: Let $\phi : X \to \Omega$ be a formula. Create a bijection between the domain of validity of ϕ and $X \times_\Omega \mathbf{1}$, the pullback of ϕ and true.

Remark 5: In fact, we did things the other way around. When working in full generality, we do no have sets, and we are forced to define domains of validity directly via pullback. So rather, Exercise 3 should be the definition, and Definition 4 should be the Exercise. However, we make things a little bit simpler for what comes next by doing it this way.

Let us see an example.

Exercise 4: Let $E : \mathbb{N} \to \Omega$ be the formula sending the number n to 1 if it is even, and to 0 else.

1. What is $\mathbb{N}_E$?
2. Wild guess: if E had to be a formula in one free variable $n \in \mathbb{N}$, what should it be?

3 Kripke-Joyal semantics

In this section, we see why formulas are called formulas: we can treat them as formulas. In topos theory, this is known as the Mitchell-Benabou language, and the Kripke-Joyal semantics give a way to talk about the truth values of a formula.

Definition 6: We call $\neg : \Omega \to \Omega$ the function sending 0 to 1 and 1 to 0. If $\phi : X \to \Omega$, we call $\neg\phi$ the function $\neg \circ \phi$. We are creating the *negation* of the formula.

Let us see how this negation behave.

Exercise 5: Let $\phi : X \to \Omega$ be a formula.

1. Prove that $X_{\neg\phi} = X \backslash (X_\phi)$.
2. Recall the formula $E : \mathbb{N} \to \Omega$ from Exercise 4. What is the domain of validity of $\neg E$?

We need a little bit more of notations before defining the $\wedge$ and the $\vee$.

Definition 7: Let $f : X \to Y$ and $f' : X \to Y'$ be two functions. We define the *product* of f and f' to be the function:

$$(f, f') : X \to Y \times Y'$$
$$x \mapsto (f(x), f'(x)).$$

Definition 8:

1. We define the *and* function $\wedge : \Omega \times \Omega \to \Omega$ to be such that $\wedge(x, y)$ is 1 if and only if $x = 1$ *and* $y = 1$ (so 0 in all other cases).
2. We define the *or* function $\wedge : \Omega \times \Omega \to \Omega$ to be such that $\vee(x, y)$ is 1 if and only if $x = 1$ *or* $y = 1$ (so 0 in all other cases).

Now, let $\phi, \psi : X \to \Omega$ be two formulas, we define

$$\phi \wedge \psi := \wedge \circ (\phi, \psi),$$

and

$$\phi \vee \psi := \vee \circ (\phi, \psi),$$

Remark 9: Compare the definition of $\neg, \wedge$ and $\vee$ with the truth tables... Can you also come up with a definition of an $\Rightarrow$ function?

Exercise 6: Let $\phi, \psi : X \to \Omega$ be two formulas.

1. Prove that $X_{\phi \wedge \psi} = X_\phi \cap X_\psi$.
2. Prove that $X_{\phi \vee \psi} = X_\phi \cup X_\psi$.

Now, we define... the truth.

Definition 10: We say that a formula $\phi : X \to \Omega$ is *true* if $X_\phi = X$. That is, a formula is true if its domain of validity is the same as the whole X. In the same spirit, we say that a formula is *false* if $X_\phi = \emptyset$.

We make precise the remark about truth in Notation 1.

Exercise 7: Let X be a set.

1. Prove that the formula $\phi_1 : X \to \Omega$ sending every $x \in X$ to 1 is true.
2. Prove that the formula $\phi_0 : X \to \Omega$ sending every $x \in X$ to 0 is false.

We prove some facts that we already knew about formulas.

Exercise 8: Let $\phi : X \to \Omega$ be a formula.

1. Prove that $\phi \vee (\neg\phi)$ is true.
2. Prove that $\phi \wedge (\neg\phi)$ is false.

Remark 11: When in a topos, the formulas $\phi \vee (\neg\phi)$ are always true, we say that the topos is boolean. Therefore, we saw in the previous exercise that the topos of sets is boolean. However, not all toposes are boolean, and this is why one must be careful when using the excluded middle (Google that).

Now we do the quantifiers.

Definition 12: Let $\phi : X \to \Omega$ be a formula. We define the formula $\forall_Y \phi : X \to \Omega$ to be the formula that sends x to 1 if and only if

$$\forall y \in Y, \phi(x, y) = 1,$$

and 0 in all other cases.
Similarly, we define $\exists_Y \phi : X \to \Omega$ to be the formula that sends x to 1 if and only if

$$\exists y \in Y, \phi(x, y) = 1,$$

and 0 in all other cases.

In fact, quantifiers are nothing but adjoints to the pullback functors along the projections, am I right? No need to understand this sentence, we can just do the next exercise (or not).

Exercise 9 (Secretly, an adjunction): Let $\pi : X \times Y \to X$ be the function sending (x, y) to x.

$$\pi^\circ : \{g : Z \to X \times Y \mid Z \text{ a set, } g \text{ a function } \} \to \{f : Z \to X \mid Z \text{ a set, } f \text{ a function } \}$$

be the function sending $g : Z \to X \times Y$ to $\pi^\circ(g) := \phi \circ g$. Let

$$\pi^* : \{f : Z \to X \mid Z \text{ a set, } f \text{ a function } \} \to \{g : Z \to X \times Y \mid Z \text{ a set, } g \text{ a function } \}$$

be the function sending $f : Z \to X$ to $\pi^*(f)$, defined to be the first projection $p_X : (X \times Y) \times_X Z \to X \times Y$ of the pullback of π and f.
Now, the exercise. Let $g : Z' \to X \times Y$ and $f : Z \to X$ be functions.

1. Prove that the set

$$R := \{h : Z \to (X \times Y) \times_X Z' \mid g = \pi^*(f) \circ h\},$$

   and the set

$$L := \{h : Z \to Z' \mid \pi^\circ(g) = f \circ h\}$$

   are in bijection (*here is the adjunction*).

2. Let $\phi : X \times Y \to \Omega$ be a formula. Prove that the set $\pi((X \times Y)_\phi)$ is in bijection with $X_{(\exists_Y \phi)}$.

Remark 13: I know, I know, some of you are worried by what we just did, Russell paradox, etc. It is gonna be alright: we can consider κ an inaccessible cardinal (Google that), and only use sets Z less than κ (or better: let us just do category theory).

We conclude.

Exercise 10: Let $\phi, \psi : X \times Y \to \Omega$ be two formulas.

1. Prove that the domain of validity of $\forall_Y(\phi \wedge \psi)$ is the same as the domain of validity of $(\forall_Y \phi) \wedge (\forall_Y \psi)$.
2. Prove that the domain of validity of $\exists_Y(\phi \vee \psi)$ is the same as the domain of validity of $(\exists_Y \phi) \vee (\exists_Y \psi)$.

Hence, conclude that:

1. $\forall_Y(\phi \wedge \psi)$ is true if and only if $(\forall_Y \phi) \wedge (\forall_Y \psi)$ is true.
2. $\exists_Y(\phi \vee \psi)$ is true if and only if $(\exists_Y \phi) \vee (\exists_Y \psi)$ is true.

Concluding remarks

Phew. Wanna be able to understand that all over again, but not with sets, just with... concepts? Then come to the category theory class next semester: we have very very good notations, and Exercise 9 + Exercise 9' (that I did not included because it is too complicated), will be encompassed in the very elegant statement:

$$\exists \dashv \pi^* \dashv \forall.$$

The quantifiers are adjoints to the pullback functor along the projection. Exercise 9 just proved $\exists \dashv \pi^*$ (and not even fully). Exercise 9 was scary, even for me: I spent one hour setting it right: this is because set-theoretical notations are not adapted for that: this is why we do category theory: it is a place were notation is wonderful. In fact, notation is so great there that then, the proof of Exercise 10 is

Solution. (**Exercise 10**)

1. $\forall$ is a right-adjoint. Right adjoints preserves limits. $\wedge$ is a limit, so $\forall$ and $\wedge$ commute.
2. $\exists$ is a left-adjoint. Left adjoints preserves colimits. $\vee$ is a colimit, so $\exists$ and $\vee$ commute.

□