Homework 1.

This homework should be manageable, but it is not telling any story.

Exercise 1: In this exercise, we permute connectives and quantifiers when we can, and give counter-examples when we cannot. Let $P(x), Q(x)$ be mathematical statement depending on a free variable $x \in X$.

1. Permuting $\forall$ and $\wedge$, both directions.
   (a) Prove that if $\forall x \in X, P(x)$ and $\forall x \in X, Q(x)$, then $\forall x \in X, P(x) \wedge Q(x)$.
   (b) Prove that if $\forall x \in X, P(x) \wedge Q(x)$, then $\forall x \in X, P(x)$ and $\forall x \in X, Q(x)$.
2. Permuting $\exists$ and $\vee$, both directions.
   (a) Prove that if $\exists x \in X, P(x)$ or $\exists x \in X, Q(x)$, then $\exists x \in X, P(x) \vee Q(x)$.
   (b) Prove that if $\exists x \in X, P(x) \vee Q(x)$, then $\exists x \in X, P(x)$ or $\exists x \in X, Q(x)$.
3. Permuting $\forall$ and $\vee$, one direction and failing.
   (a) Prove that if $\forall x \in X, P(x)$ or $\forall x \in X, Q(x)$, then $\forall x \in X, P(x) \vee Q(x)$.
   (b) Let $E(n)$ means "n is even" and $O(n)$ means "n is odd". Prove that $\forall n \in \mathbb{N}, E(n) \vee O(n)$, but that it is not the case that $\forall n \in \mathbb{N}, E(n)$ or $\forall n \in \mathbb{N}, O(n)$
4. Permuting $\exists$ and $\wedge$, one direction and failing.
   (a) Prove that if $\exists x \in X, P(x) \wedge Q(x)$, then $\exists x \in X, P(x)$ and $\exists x \in X, Q(x)$.
   (b) Let $O(n)$ means "$n = 1$ " and $T(n)$ means "$n = 2$". Prove that $\exists n \in \mathbb{N}, O(n)$ and $\exists n \in \mathbb{N}, T(n)$, but that it is not the case that $\exists n \in \mathbb{N}, O(n) \wedge T(n)$.
5. Permuting $\exists$ and $\forall$, one direction and failing. Let $R(x, y)$ be a mathematical statement depending on a variable $x \in X$ and a variable $y \in Y$.
   (a) Prove that if $\exists x \in X, \forall y \in Y, R(x, y)$, then $\forall y \in Y, \exists x \in X, R(x, y)$.
   (b) Let X be a set with two distincts elements. Let $R(x, x')$ be the statement saying "$x = x'$". Prove that $\forall x \in X, \exists x' \in X, R(x, x')$, but that it is not the case that $\exists x \in X, \forall x' \in X, R(x, x')$.

Remark 1: The real reasons why we can sometimes permute, and sometimes not, can be found in the unmanageable homework.

Exercise 2: Let X be a set, and let $A \subseteq X$.

1. Prove that $X = A \cup (X \backslash A)$, and that $A \cap (X \backslash A) = \emptyset$.

We can do the exclusive-or on sets too:

Definition 2: Let X, Y be two sets. We define $X \oplus Y$ to be the set

$$X \oplus Y := (X \backslash Y) \cup (Y \backslash X).$$

Exercise 3: Let X, Y, Z be sets.

1. Prove that $X \oplus \emptyset = X$.
2. Prove that $X \cap (Y \oplus Z) = (X \cap Z) \oplus (X \cap Y)$.
3. Prove that $X \oplus Y = \emptyset$ if and only if $X = Y$.

Exercise 4: Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that

1. If $g \circ f$ is injective, then f is injective.
2. If $g \circ f$ is surjective, then g is surjective.

3. If $g \circ f$ is bijective, then f is injective and g is surjective.

Exercise 5: For the following functions, if you think they are (injective—surjective—bijective), prove it. If you think they are not (injective—surjective—bijective), prove it too.

1.
$$\begin{aligned} f : \mathbb{N} &\to \mathbb{N} \times \mathbb{N} \\ n &\mapsto (n, n) \end{aligned}$$

2.
$$\begin{aligned} f : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (n, m) &\mapsto n \end{aligned}$$

3.
$$\begin{aligned} f : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \times \mathbb{N} \\ (n, m) &\mapsto (m, n). \end{aligned}$$

Remark: by (injective—surjective—bijective), we mean that you have to do the work for injective, surjective, and bijective.

Exercise 6: Let X be a set.

1. Find an equivalence relation $\sim$ on X such that $X/_\sim$ is
$$\{\{x\} \mid x \in X\}.$$

2. Find an equivalence relation $\sim$ on X such that $X/_\sim$ is
$$\{X\}.$$