Diagrammatic (∞, n)-categories

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Atoms and molecules

A molecule is combinatorial data encoding the shape of a higher categorical pasting diagram. A molecule U:

- is an oriented and graded poset;
- has for each $\alpha \in \{-, +\}$ and $n \in \mathbb{N}$, an input/output n-boundary, $\partial_n^\alpha U$, which is a closed subset of U.

Molecules can be pasted together: given U, V, and $k \in \mathbb{N}$, if $\partial_k^+ U \cong \partial_k^- V$, then the pushout

$$\begin{tikzcd} \partial_k^+ U \cong \partial_k^- V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ U \arrow[r, hook] & U \#_k V \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a molecule. Pasting and boundary of molecules satisfy all the axioms of strict ω-categories, and even some higher-exchange laws in dimension ≥ 4. An atom is a molecule with a greatest cell.

Directed complexes

A directed complex X is a presheaf over the category of atoms. It is the combinatorial data used to construct a positive and regular polygraphs:

$$\begin{tikzcd} \coprod_{u \in \mathscr{S}_n} \partial U \arrow[r, hook] \arrow[d, "(\partial u)_{u \in \mathscr{S}_n}"'] & \coprod_{u \in \mathscr{S}_n} U \arrow[d, "(u)_{u \in \mathscr{S}_n}"] \\ X_{n-1} \arrow[r, hook] & X_n, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

There is a construction Mol/X that produces the strict ω-category of molecules in X.

- If X is a graph, Mol/X is the free category on X;
- If X is the n-dimensional simplex, then Mol/X is the n-oriental.

Theorem 1. Under a general acyclicity condition on its molecules, Mol/X is a polygraph, and in fact, a weak polygraph.

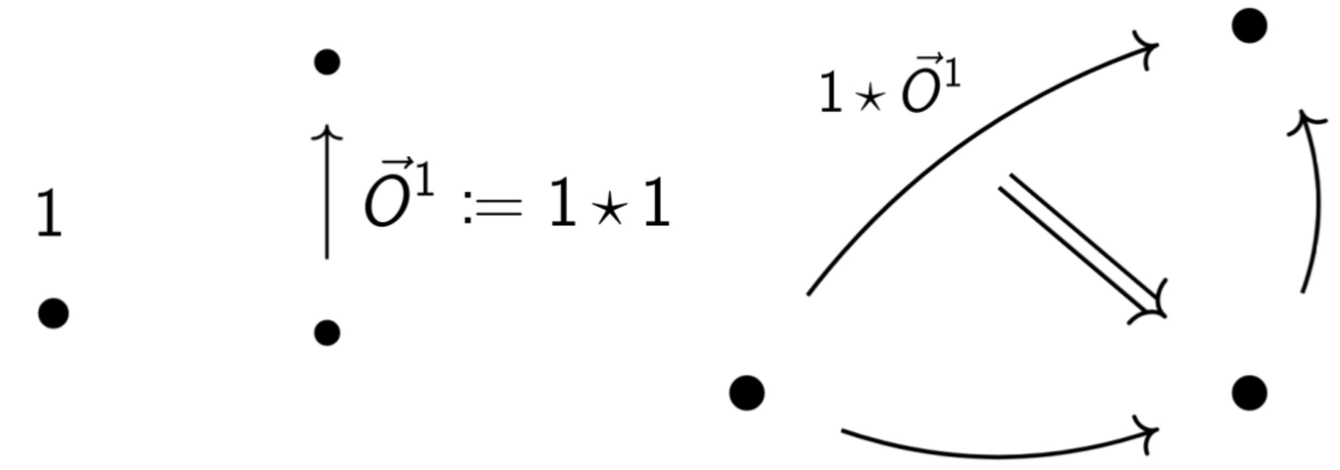
Operations

Atoms, molecules, and directed complexes, natively support higher categorical operations:

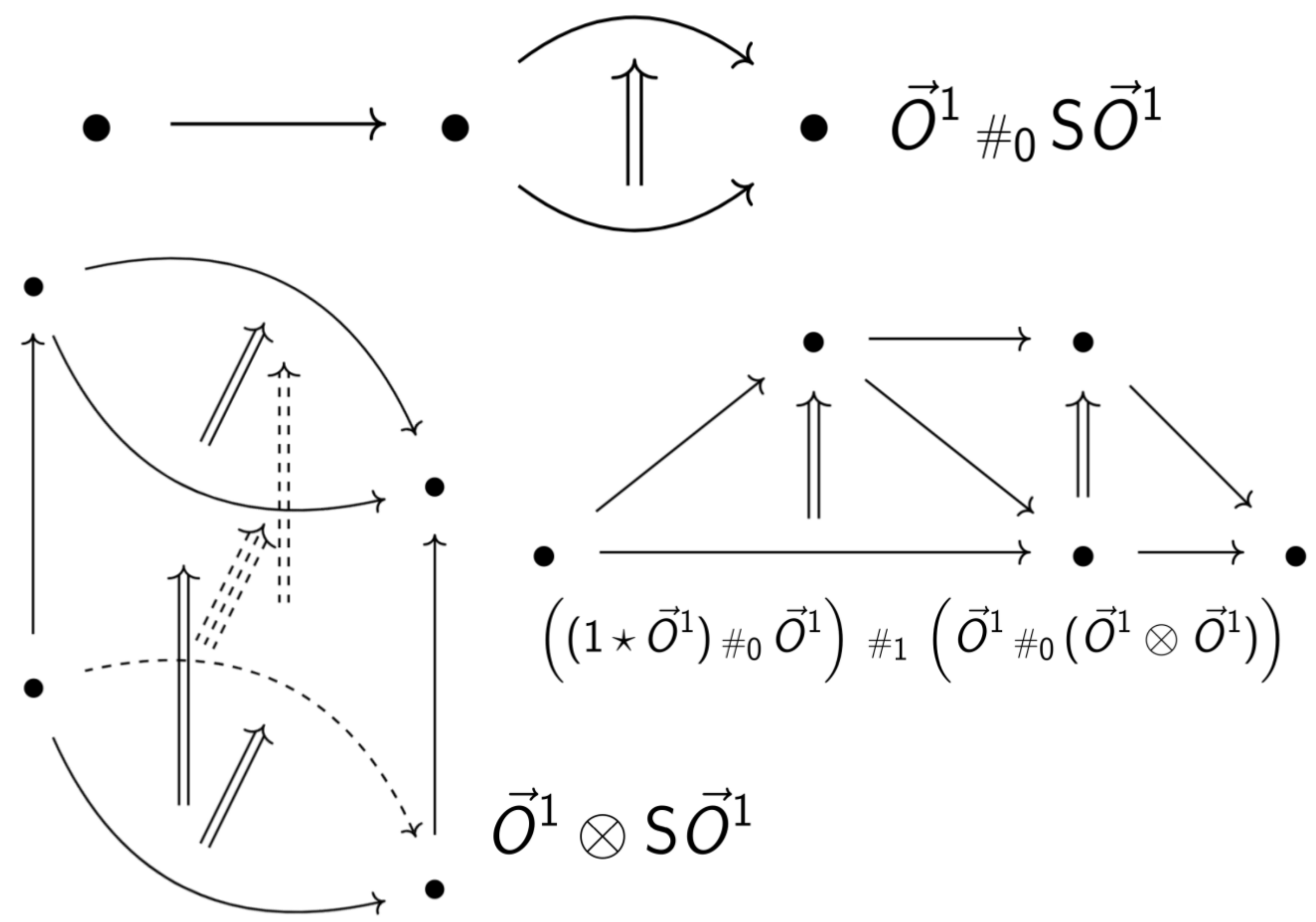
- Gray product $X \otimes Y$,
- join $X \star Y$,
- suspension $\mathsf{S}X$,
- duals $\mathsf{D}_J X$, where $J \subseteq \mathbb{N}$.

Examples

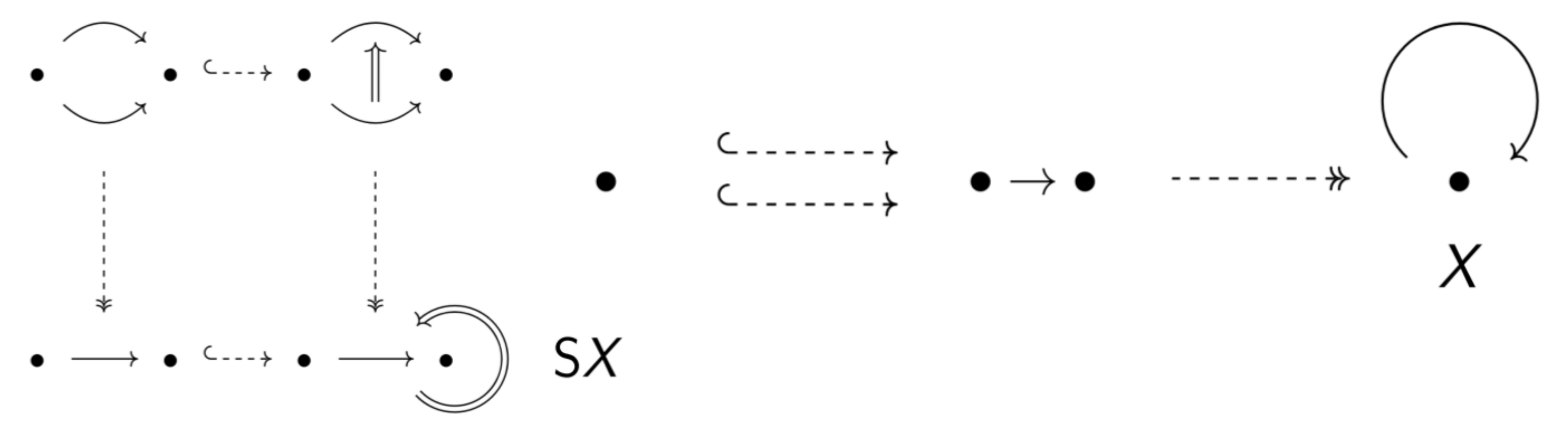
Atoms



Molecules



Directed complexes



In the last example:

- $Mol/X = \mathsf{B}\mathbb{N}$
- X and $\mathsf{S}X$ satisfy the acyclicity hypothesis of **Theorem 1.**

Diagrammatic (∞, n)-categories

Directed complexes may be used in several ways to construct models of strict n-categories and (∞, n)-categories:

1. as Segal sheaves over the Mol/X's and strict functofdatrs;
2. as presheaves over the atoms and cellular strict functors between them. A localisation enforces the existence of weak composites.

This matches the following two perspectives on standard (∞, n)-categories as

- Segal sheaves over the thetas and strict functors;
- presheaves over the (marked) orientals and cellular strict functors (i.e. over the marked simplex category).

Theorem. Point 1. is a localisation of the (∞, n)-categories at the higher-exchanges, which does nothing when $n \leq 3$.

Conjecture. Point 1. and 2. present the same model of higher categories.

Features of the diagrammatic models

- Very strong pasting theorems;
- Semi-strictification: marked directed complexes may be used to produce two equivalent model:
  1. one where composition is witnessed by the existence of weak composites;
  2. one where composition is an operation satisfying strict associativity and exchange;

  In both cases, the units are algebraic and weak.
- Native support for higher categorical constructions, with no need to use different presentations.
- A rich algebra of weak units.
- A practical formalism that feels like working with a topologically sound alternative to strict n-categories

References

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