# Gray products (of diagrammatic $(\infty, n)$-categories)

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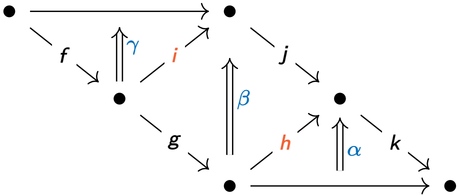
# Generalised pasting

*Generalised pasting* is a new technology

1. introduced by Hadzihasanovic in his book *Combinatorics of higher-categorical diagrams* [Had24];
2. that allows you to create (higher) *composable diagrams* by gluing along *portions* of boundaries of cells;

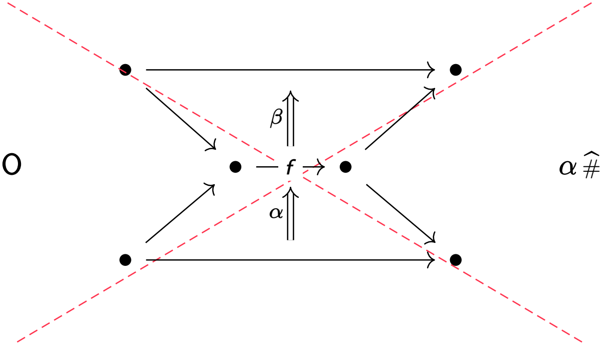



| composing *without* generalised pasting | composing *with* generalised pasting |
|---|---|
| $(k * j * \gamma) \circ (k * \beta * f) \circ (\alpha * g * f)$ | $\alpha \,\widehat{\#}_h\, \beta \,\widehat{\#}_i\, \gamma$ |

The *generalised pasting* deal also comprises

1. a *no-nonsense* guarantee, preventing you from creating silly non-composable diagrams:

NO




$\alpha \,\widehat{\#}_{f}\, \beta$

2. a generalisation usual pasting and *grafting*;
3. and, of course, *a distribution over Gray products*.

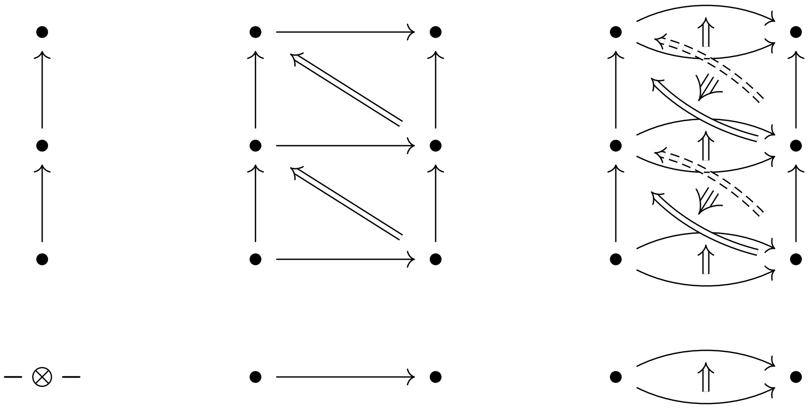
# Gray products

# Facts

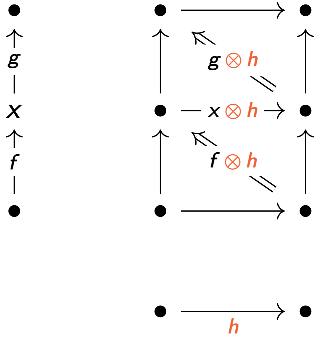
The *Gray product*:

- *semicartesian* monoidal product (for a variety of higher structures);
- usually written $- \otimes -$;
- *not* symmetric;
- adds the dimension of its inputs: $\dim(f \otimes g) = \dim f + \dim g$.
- useful for modelling all sorts of (co)lax transformations.

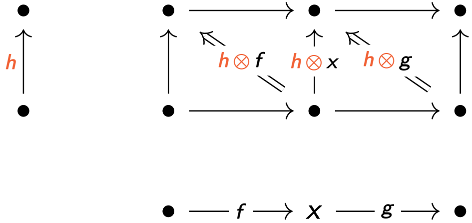
# Pictures



# Formulas




$$(f \mathbin{\widehat{\#}_x} g) \otimes h = (f \otimes h) \mathbin{\widehat{\#}_{x \otimes h}} (g \otimes h)$$




$$h \otimes (f \mathbin{\widehat{\#}_x} g) = \begin{cases} (h \otimes f) \mathbin{\widehat{\#}_{h \otimes x}} (h \otimes g) & \dim h \text{ is } \textit{even}, \\ (h \otimes g) \mathbin{\widehat{\#}_{h \otimes x}} (h \otimes f) & \dim h \text{ is } \textit{odd}. \end{cases}$$

# Tensoring with an equivalence

**Theorem 1:** *Let $\mathcal{C}, \mathcal{D}$ be higher structures, let*

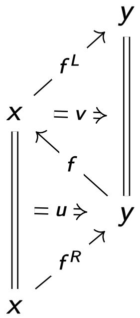
$$f \text{ be a } n\text{-cell in } \mathcal{C}, \qquad g \text{ be a } p\text{-cell in } \mathcal{D}.$$

*Then $f \otimes g$ is invertible in $\mathcal{C} \otimes \mathcal{D}$ as soon as $f$* ***or*** *$g$ is.*

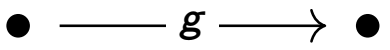
Say $f$ is invertible. How to compute $(f \otimes g)^*$ in function of:

$$f \otimes g, \qquad f^* \otimes g, \quad \ldots?$$

*I do not know, yet the Theorem is proven in* [Cha25].




imagine the cube here




- *have:* $f$ invertible, i.e

$$u \colon 1_x \simeq f^R \,\widehat{\#}_y\, f \quad \text{and} \quad v \colon f \,\widehat{\#}_x\, f^L \simeq 1_y.$$

- *want:* $f \otimes g$ invertible;
- *assume coinductively* that $u \otimes g$ and $v \otimes g$ are invertible;
- *deduce* that

$$(f^R \,\widehat{\#}\, f) \otimes g = (f^R \otimes g) \,\widehat{\#}\, (f \otimes g) \qquad \text{and}$$
$$(f \,\widehat{\#}\, f^L) \otimes g = (f \otimes g) \,\widehat{\#}\, (f^L \otimes g)$$

  are invertible.
- *use* fancy 2-out-of-6 for generalised pasting:

$$\overbrace{(f^R \otimes g) \,\widehat{\#}\, (f \otimes g)}^{\text{invertible}} \,\widehat{\#}\, (f^L \otimes g) \quad \text{with} \quad \underbrace{(f \otimes g) \,\widehat{\#}\, (f^L \otimes g)}_{\text{invertible}}.$$

# References I


- Clémence Chanavat, *Gray products of diagrammatic* $(\infty, n)$*-categories*, 2025, arXiv:2505.01387.
- A. Hadzihasanovic, *Combinatorics of higher-categorical diagrams*, Online preprint arXiv:2404.07273v2, 2024, To appear in London Mathematical Society Lecture Note Series.

Thanks!