Diagrammatic sets as a model of homotopy types

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Deconstructiong the title

Diagrammatic sets	*the presheaves over the shapes of this morning*
as a model	*have a model structure which is Quillen equivalent*
of homotopy types	*to "the one" on topological spaces*

Structure of the talk

1. Brief reminder of model structures.
2. Model structures on categories of presheaves according to Cisinski.
3. Diagrammatic sets.
4. Application: Model structure on diagrammatic sets.

Model structures

Localisation

Let $\mathcal{C}$ be a category. Let W be a class of maps in $\mathcal{C}$ such that it would be nice if they were isomorphisms.

Definition 1.1: The *localisation* of $\mathcal{C}$ at W is the best category $\mathcal{C}[W^{-}1]$ pretending that all maps in W are isomorphisms.

Construction: formally add all the inverses on the underlying graph; take the free category; quotient by the appropriate compositions.
Problem: it always exists but is very hard to compute in practice.

Model structures

Let $\mathcal{C}$ be a (co)complete category. A *model structure* is the data of three classes of maps:

1. The weak equivalences[2] W: *the maps we want to invert*;
2. The cofibrations C: *the accompanying "nice injections"*;
3. The fibrations F: *the accompanying "nice surjections"*;

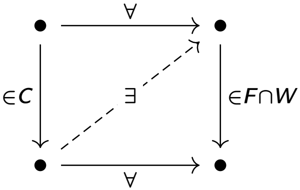
Such that:

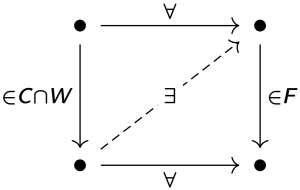
$$(C, F \cap W) \qquad \text{and} \qquad (C \cap W, F)$$

are *weak factorisation systems*.

[2] contain iso, and have 2-out-of-3

The squares to keep in mind





Localisation and Quillen adjunctions

Let $(\mathcal{C}, W, C, F)$, $(\mathcal{D}, W', C', F')$ be a model categories.

- We can form the localisation $\mathcal{C}[W^{-1}]$ and $\mathcal{D}[W'^{-1}]$.
- Let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor. *When does it induce a functor*

$$\mathbb{L}\colon \mathcal{C}[W^{-1}] \to \mathcal{D}[W'^{-1}]?$$

  Suffices: L left adjoint, and preserves cofibrations and acyclic cofibrations.
- We call it a *left Quillen functor*.
- If $\mathbb{L}$ is an equivalence: *Quillen equivalence*.

Examples

	weak equivalences	cofibrations	acyclic cofibrations
Top	homotopy equivalences	(retract of) relative cell complexes	(generated by) $D^n \hookrightarrow [0,1] \times D^n$
sSet	combinatorial homotopy equivalences[3]	(retract of) combinatorial relative cell complexes[4]	(generated by) $\Lambda^k_n \hookrightarrow \Delta^n$

The geometric realisation

$$|-|: \mathbf{sSet} \to \mathbf{Top}$$

is a left Quillen equivalence.

Title of this talk 57 years ago:

Simplicial sets as a model of homotopy types.

[3] only true between fibrant object

[4] i.e. monomorphisms

Aparté: ∞-groupoids

Definition 1.2: A simplicial set X is an *∞-groupoid* if

$$\begin{array}{ccc} \Lambda^k_n & \xrightarrow{\forall} & X \\ \downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

that is, $X \to \mathbf{1}$ is a fibration in the model structure!

Theorem 1.1: *Up to homotopy, ∞-groupoids and topological spaces are the same.*

Cisinski's model structures

Cisinski's model structure

Let $\mathcal{C}$ be a (small) category.

- *Question:* How to put a model structure on $\widehat{\mathcal{C}} := [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$?
- *Answer:* Just give a *homotopical structure*!

It consists of two things:

1. An *exact cylinder to define notion of homotopy*,
2. A class of *anodyne extensions to extend homotopies*.

Cylinder

A *cylinder* is an endofunctor

$$\mathsf{I}\colon \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$$

with three natural transformations:

- $\iota^\alpha\colon \mathrm{id}_{\widehat{\mathcal{C}}} \to \mathsf{I}$, $\alpha \in \{-,+\}$ *the two sections*
- $\sigma\colon \mathsf{I} \to \mathrm{id}_{\widehat{\mathcal{C}}}$ *the projection*

Such that for all $\alpha \in \{-,+\}$, $X \in \widehat{\mathcal{C}}$:

$$X \xrightarrow{\iota^\alpha_X} \mathsf{I}X \xrightarrow{\sigma_X} X, \qquad \sigma_X \circ \iota^\alpha_X = \mathrm{id}_X$$

Exact cylinder

A cylinder $(\mathsf{I}, \iota^\alpha, \sigma)$ is *exact* if furthermore:

- I preserves small colimits and monomorphisms;
- some square is a pullback.

Example 2.1: *The endofunctor* $[0,1] \times - \colon \mathbf{Top} \to \mathbf{Top}$ *together with, for all* $X \in \mathbf{Top}$,

$$\iota^-_X \colon X \hookrightarrow [0,1] \times X \qquad \iota^+_X \colon X \hookrightarrow [0,1] \times X$$
$$x \mapsto (0,x) \qquad\qquad x \mapsto (1,x)$$

and

$$\sigma_X \colon [0,1] \times X \twoheadrightarrow X$$
$$(i,x) \mapsto x$$

is an exact cylinder[a].

[a] if we pretend that **Top** is a category of presheaves

Homotopy

Let $f, g \colon X \to Y$ be two natural transformations in $\widehat{\mathcal{C}}$, a *homotopy* from f to g is a map $\phi \colon \mathsf{I}X \to Y$ such that

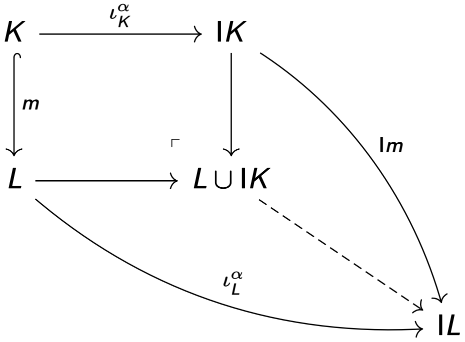
$$\begin{array}{ccccc} X & \xrightarrow{\iota_X^-} & \mathsf{I}X & \xleftarrow{\iota_X^+} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle g} & \\ & & Y & & \end{array}$$

Anodyne extensions

A class of maps **An** is called class of *anodyne extensions* if

AN0 It is generated by a set;

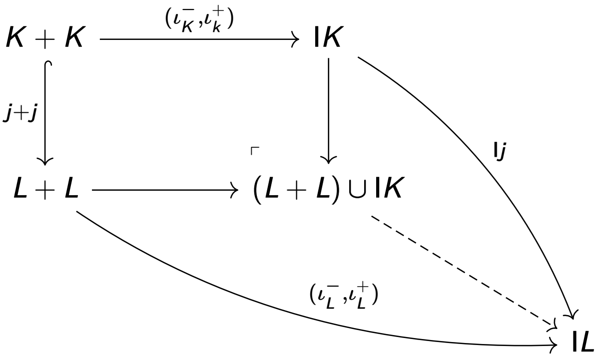
AN1 For all monomorphisms $m\colon K \hookrightarrow L$ in $\widehat{\mathcal{C}}$, for all $\alpha \in \{-, +\}$, the dashed map



is in **An**. This is the *cylinder filling property;*

Anodyne extensions

AN2 For all $j\colon K \hookrightarrow L \in \mathbf{An}$, the dashed map



is in $\mathbf{An}$. This is the *equivalence of the cylinder fillings.*

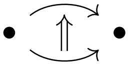
With that... [Cis06]

There is a model structure on $\widehat{\mathcal{C}}$ where

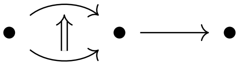
- cofibrations $=$ monomorphisms,
- anodyne extensions $\subseteq$ acyclic cofibrations,
- an object is fibrant iff it has the r.l.p. against anodyne extensions,
- weak equivalences between fibrant objects are exactly the homotopy equivalences.

Interlude

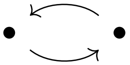
Which is which?



atom



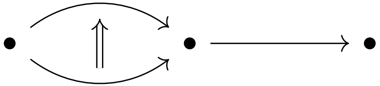
regular directed complex



molecule

Boundaries

Let U be the molecule:



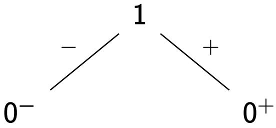
What is:

- $\partial_1^- U$?
- $\partial_0^+ U$?

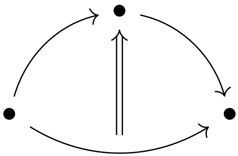
Is it round?

Gray product

Let $\vec{I}$ be $\bullet \to \bullet$, i.e. the ogpos



Draw

$\vec{I} \quad \otimes$ 

Diagrammatic sets

Maps of regular directed complexes

Let P, Q be **RDCpx**. A *map* $f\colon P \to Q$ is

- an order-preserving map f of the underlying posets,
- for all $n \geq 0$, $\alpha \in \{-, +\}$, $x \in P$

$$f(\partial_n^\alpha x) = \partial_n^\alpha f(x),$$

- some finality conditions.

Observe: *injections are nice, surjections are not.*
Idea: impose a further condition.

- *which is aslo a Grothendieck fibration of the underlying posets*.

(aka *f is cartesian*)

More on cartesian maps

- Injections were already cartesian.
- Less surjections.
- $\otimes, \star$ are still monoidal!

Eilenberg–Zilber category

Theorem 4.1: *The category $\odot$ with cartesian map is an Eilenberg–Zilber category.*

In particular:

1. epi-mono factorisation,
2. no non-identity automorphism,
3. *each surjection has a section, and is entirely determined by them!*

Diagrammatic set

Definition 4.1: A *diagrammatic set* is a presheaf on $\odot$.

Proposition 4.1: *The Yoneda embedding* $\odot \hookrightarrow \odot\mathbf{Set}$ *factors through:*

$$\odot \hookrightarrow \textit{molecules} \hookrightarrow \mathbf{RDCpx} \hookrightarrow \odot\mathbf{Set}$$

Building a cylinder

Day convolution

"Recall": let $(\mathcal{C}, \otimes, \mathbf{1})$ monoidal, we can make $\widehat{\mathcal{C}}$ into a monoidal category $(\widehat{\mathcal{C}}, \otimes, \mathbf{1})$ such that:

- Yoneda $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ is strong monoidal,
- $- \otimes -$ is biclosed.

If $X, Y \in \widehat{\mathcal{C}}$, then

$$X \otimes Y = \int^{c,d \in \mathcal{C}} X(c) \times Y(d) \times \mathcal{C}(-, c \otimes d).$$

Since $(\odot, \otimes, \mathbf{1})$ is monoidal, we can take Gray product of presheaves!

Cylinder object

So we define:

$$\vec{I} \otimes - : \odot\mathbf{Set} \to \odot\mathbf{Set}$$

And factoring $\mathrm{id} : \mathbf{1} \to \mathbf{1}$, $\alpha \in \{-, +\}$ as

$$\mathbf{1} \overset{\iota^\alpha}{\hookrightarrow} \vec{I} \overset{!}{\twoheadrightarrow} \mathbf{1}$$

we have, for all $X \in \odot\mathbf{Set}$,

$$X \overset{\iota^\alpha_X}{\hookrightarrow} \vec{I} \otimes X \overset{\sigma_X}{\twoheadrightarrow} X$$

$\vec{I} \otimes -$ is a cylinder

Exact cylinder object

Lemma 5.1: *The cylinder* $\vec{I} \otimes -$ *is exact.*

An immediate consequence

A corollary of [Cis06, Corollaire 8.2.16]:

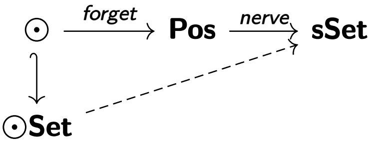
Corollary 5.1: *Let $\mathcal{C}$ be an Eilenberg–Zilber category with a terminal object. Let $(\mathsf{I}, \iota^{\alpha}, \sigma)$ be an exact cylinder. Suppose that for all $c \in C$, the map*

$$\sigma_c : \mathsf{I}c \twoheadrightarrow c$$

is representable.
Then $\widehat{\mathcal{C}}$ models homotopy types.

As a corollary of the corollary of [Cis06, Corollaire 8.2.16]:

Theorem 5.1: *Diagrammatic sets model homotopy types. There is a model structure on* $\odot$**Set** *whose homotopy category are the homotopy types. The Quillen equivalence with* **sSet** *is given by the following left Kan extension, called the diagrammatic subdivion:*



What is this model structure?

Horns

Let U be an atom, let $V \sqsubseteq \partial^\alpha U$ be a rewritable submolecule of the input or output boundary. The *horn* defined by U, V is

$$\Lambda_U^V := \partial U \setminus (V \setminus \partial V).$$

Let

$$J := \{\lambda_U^V : \Lambda_U^V \hookrightarrow U\}.$$

Horns generate anodyne extensions

Proposition 6.1: *The set J of horns generates a class of anodyne extensions.*

Proof.

AN0 by definition.

AN1 by picture.

AN2 by picture.

□

Another model structure

So *there is another model structure on* $\odot$**Set**...

- which is seen to be monoidal for $- \otimes -$;
- whose acyclic cofibrations are exactly the anodyne extensions!

Theorem 6.1: *The two model structures are the same.*

Aparté: back to ∞-groupoids

Definition 6.1: A diagrammatic set X is an *∞-groupoid* if

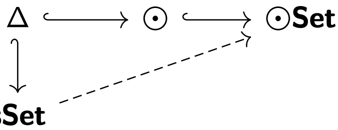
$$\begin{array}{ccc} \Lambda_U^V & \xrightarrow{\forall} & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ U & & \end{array}$$

that is, $X \to \mathbf{1}$ is a fibration in the model structure!

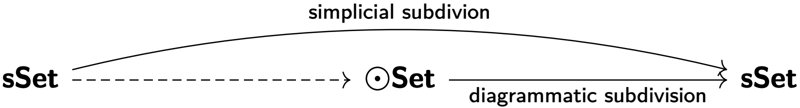
Theorem 6.2: *Up to homotopy, ∞-groupoids and topological spaces are the same.*

Bonus: another Quillen equivalence

The following left Kan extension



is a left Quillen equivalence! Proof:



useful to prove that the cartesian product of diagrammatic sets is homotopical

Conclusion

- $\odot$ is a well behaved category of shapes, with a representable cylinder;
- The model structure that comes automatically from this models homotopy types;
- With a further study, we find
  - acyclic cofibrations are generated by the horns;
  - the Gray product is monoidal;
  - there are two Quillen equivalences with simplicial sets

$$\mathbf{sSet} \leftrightarrows \odot\mathbf{Set} \leftrightarrows \mathbf{sSet}$$

    factorising the Quillen equivalence $\mathrm{Sd} \dashv \mathrm{Ex}$

References I

D.-C. Cisinski, *Les préfaisceaux comme modèles des types d'homotopie*, Astérisque, no. 308, Société mathématique de France, Paris, 2006 (fre).

Thanks!