Pasting theorems

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Us, category theorists

We, category theorists, speak the language of diagrams.
Today, I will talk about diagrams of a specific kind:

pasting diagrams.

Them, set theorists

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From our first commutative diagrams

$$
\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\
Fc' \arrow[r, "\alpha_{c'}"'] & Gc'
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & (u \otimes v) \otimes (w \otimes x) \arrow[drr, "\alpha_{u\otimes v,w,x}"] & & \\
u \otimes (v \otimes (w \otimes x)) \arrow[urr, "1_u\otimes\alpha_{v,w,x}"] \arrow[dr, "\alpha_{u,v,w\otimes x}"'] & & & & ((u \otimes v) \otimes w) \otimes x \\
& u \otimes ((v \otimes w) \otimes x) \arrow[rr, "\alpha_{u,v\otimes w,x}"'] & & (u \otimes (v \otimes w)) \otimes x \arrow[ur, "\alpha_{u,v,w}\otimes 1_x"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, phantom, "\Downarrow\gamma"] & \mathsf{E} \\
& \mathsf{D} \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & \mathsf{E} \\
& \mathsf{D} \arrow[ur, bend left, "\mathrm{Lan}_K F"] \arrow[ur, bend right=50, "G"'] \arrow[ur, phantom, "\exists!\Searrow"] &
\end{tikzcd}
$$

Source: *Category theory in context*, Riehl

To more elaborate ones

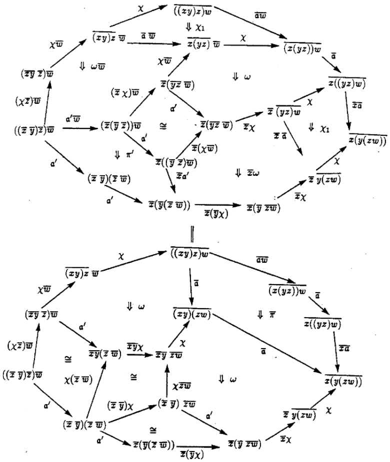
$$
\begin{tikzcd}
FX \arrow[r, "Fg"] \arrow[dr, "F(hg)"] \arrow[dr, phantom, shift right=4ex, "F^2\Downarrow"] & FY \arrow[d, "Fh"] \arrow[dl, phantom, shift left=2ex, "F^2 \Swarrow"] \\
FW \arrow[u, "Ff"] \arrow[r, "F((hg)f)"] \arrow[r, bend right=50, "F(h(gf))"'] \arrow[r, phantom, shift right=3ex, "Fa \Downarrow"] & FZ
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, "Fg"] \arrow[dr, phantom, shift right=2ex, "F^2 \Searrow"] & FY \arrow[d, "Fh"] \\
FW \arrow[u, "Ff"] \arrow[ur, "F(gf)"'] \arrow[r, bend right=50, "F(h(gf))"'] \arrow[r, phantom, "\Downarrow F^2"] & FZ
\end{tikzcd}
$$

Icon Naturality:

$$
\begin{tikzcd}
& FY \arrow[dr, bend left, "Fg"] \arrow[d, phantom, "\Downarrow F^2"] & \\
FX \arrow[ur, bend left, "Ff"] \arrow[rr, "F(gf)"] \arrow[rr, bend right=40, "G(gf)"'] \arrow[rr, phantom, shift right=3ex, "\Downarrow \alpha_{gf}"] & & FZ
\end{tikzcd}
=
\begin{tikzcd}
& FY \arrow[dr, bend left, "Fg"] \arrow[dr, "Gg"'] \arrow[dr, phantom, shift left=2ex, "\alpha_g \Swarrow"] & \\
FX \arrow[ur, bend left, "Ff"] \arrow[ur, "Gf"'] \arrow[ur, phantom, shift left=2ex, "\alpha_f \Searrow"] \arrow[rr, bend right=40, "G(gf)"'] \arrow[rr, phantom, "\Downarrow G^2"] & & FZ
\end{tikzcd}
\qquad (4.6.4)
$$

Source: *2-Dimensional Categories*, Johnson and Yau

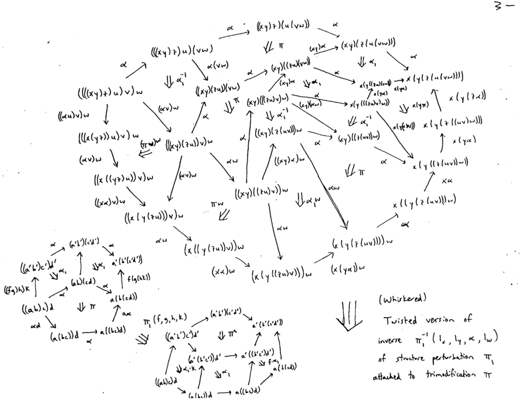
or everyone's favorite paper. . .



Source: *Coherence for tricategories*, GPS

and Trimble's tetracategories

"it is recommended that the reader print them out (large) and spread them out on a wide surface, and arrange them in a circular pattern"



Source: `https://math.ucr.edu/home/baez/trimble/tetracategorical_data.pdf`

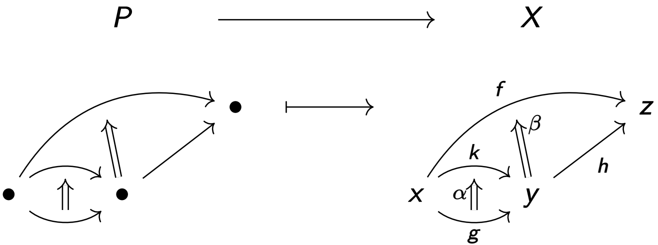
An open question

What gives us the right?

The answer lies in the colimits.

A *pasting diagram* is a higher categorical structure freely generated from underlying combinatorial data.

Let P be a pasting diagram and X be a higher structure.
A diagram of shape P in X is a functor $P \to X$.



Since P is a free higher structure on its underlying data, a functor is a compatible family of cells

$$\left\{ s_x \colon \vec{O}^{\dim x} \to X \right\}_{x \in P}.$$

Yes, but. . .

P is a free strict higher structure,

$$\text{data structure} \xrightarrow[\text{free functor}]{\mathrm{F}} \text{strict world} \xhookrightarrow[\text{inclusion (right adjoint)}]{\iota} \text{weak world}$$

$$P \cong \operatorname*{colim}_{x \in P} P_x \qquad \mathrm{F}P \cong \operatorname*{colim}_{x \in P} \mathrm{F}P_x \qquad \iota\mathrm{F}P \overset{?}{\cong} \operatorname*{colim}_{x \in P} \iota\mathrm{F}P_x$$

Definition 1.1: A *pasting theorem* gives sufficient conditions guaranteeing that the inclusion ι preserves certain colimits.

Colimits: what do they know and how to compute them?

The three realms of homotopy coherent mathematics

realm:	strict		comp		weak
	$\mathscr{C}_0$	$\xhookrightarrow[\iota]{}$	$\mathscr{C}$	$\xleftrightarrow{\text{proxy}}$	$\mathscr{C}_\infty$
	$\underline{\mathrm{Cat}}_{(0,n)}$	$\xhookrightarrow[\iota]{}$	$\mathrm{Psh}(\mathbf{dgm})$	$\xleftrightarrow{\text{proxy}}$	$\underline{\mathrm{Cat}}_{(\infty,n)}$

have: colimits in the strict realm $\mathscr{C}_0$.

want: colimits in the weak realm $\mathscr{C}_\infty$.

can: colimits in the computational realm $\mathscr{C}$.

Concrete experiments

In the next three examples, we place ourselves in the following instance of the realms:

$$\underline{\mathrm{Cat}} \overset{\iota}{\hookrightarrow} \underline{\mathrm{sSet}} \overset{\text{proxy}}{\dashleftarrow\dashrightarrow} \underline{\mathrm{Cat}}_{(\infty,1)}.$$

1. The Composable Arrows

Consider the following colimit:

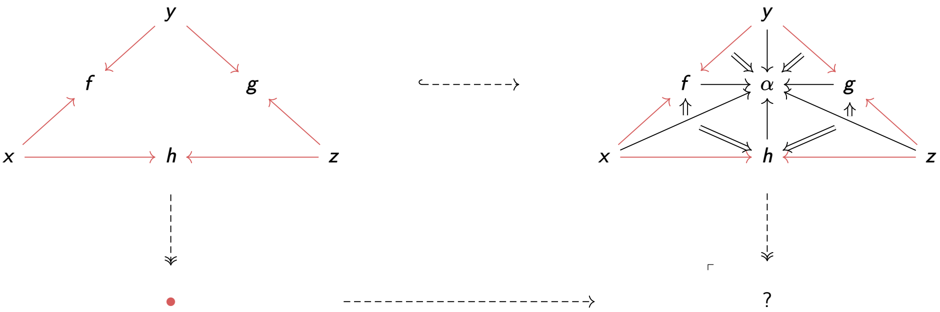
$$\begin{tikzcd}
\bullet \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \bullet \xrightarrow{g} \bullet \arrow[d, hook, dashed] \\
\bullet \xrightarrow{f} \bullet \arrow[r, hook, dashed] & ? \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

computed in $\underline{\mathrm{sSet}}$ versus computed in $\underline{\mathrm{Cat}}$ or $\underline{\mathrm{Cat}}_{(\infty,1)}$

$$\begin{tikzcd}
& \bullet \arrow[dr, "g"] & & & & \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] & & \bullet \arrow[r, hook, dashed, "\text{comparison}"'] & & \bullet \arrow[ur, "f"] \arrow[rr, "gf"'] & \arrow[u, phantom, "\Uparrow"] & \bullet
\end{tikzcd}$$

2. The Sphere or the Arrow

Consider the following colimit:



When computed in

- $\underline{\mathrm{Cat}}$, it is the arrow category $\bullet \to \bullet$, but
- $\underline{\mathrm{sSet}}$ and $\underline{\mathrm{Cat}}_{(\infty,1)}$, it is something whose underlying space is a 2-sphere.

3. The Point or the Circle

Consider the following colimit

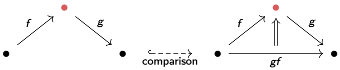
$$\begin{tikzcd}
\bullet \quad \bullet \arrow[r, dashed, two heads] \arrow[d, dashed, two heads] & \bullet \arrow[d, dashed, two heads] \\
\bullet \arrow[r, dashed, two heads] & ? \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

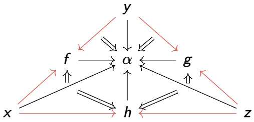
When computed in

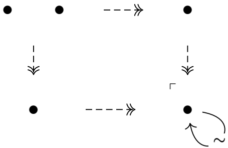
- $\underline{\mathrm{Cat}}$ and $\underline{\mathrm{sSet}}$, it is the point $\bullet$,
- $\underline{\mathrm{Cat}}_{(\infty,1)}$, it is something whose underlying space is a circle.

Who is right? Who is wrong?

1. Composable arrows:
   - $\underline{\mathrm{Cat}}$ is right;
   - $\underline{\mathrm{sSet}}$ is right;
   - $\underline{\mathrm{Cat}}_{(\infty,1)}$ is right;
2. Sphere or arrow:
   - $\underline{\mathrm{Cat}}$ is wrong;
   - $\underline{\mathrm{sSet}}$ is right;
   - $\underline{\mathrm{Cat}}_{(\infty,1)}$ is right;
3. A circle, not a point
   - $\underline{\mathrm{Cat}}$ is wrong;
   - $\underline{\mathrm{sSet}}$ is wrong;
   - $\underline{\mathrm{Cat}}_{(\infty,1)}$ is right;







homotopy is always right.

Summary

- computing (co)limits in homotopy coherent structures via the computational proxy is fairly well-understood: it is the theory of *homotopy (co)limits*.
- it is harder to know whether a colimit of strict structures is also a colimit in the weak sense, and is less well-understood.
- a *pasting theorem* gives guarantee that certain colimits in the strict realm are also colimits in the weak realm.

Pasting theorems for (∞, n)-categories

Some definitions

An *atom* is a pasting diagram with a greatest element:

$$\bullet \qquad\qquad \bullet \longrightarrow \bullet$$

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\Uparrow"] & \bullet \\
& \bullet \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Uparrow\!\Rrightarrow\!\Uparrow"] & & \bullet
\end{tikzcd}$$

$$\ldots$$

A *directed complex* is any formal colimit of atoms. We let $\underline{\mathrm{dCpx}}$ be the category of directed complexes.

There is a functor $\underline{\mathrm{dCpx}} \to \underline{\mathrm{Cat}}_{(0,\omega)}$ that generates the free[1] $(0, \omega)$-category on a directed complex.

We say that a directed complex is a *homotopy polygraph* if this free $(0, \omega)$-category is also a free (∞, ω)-category.

[1]it depends, actually.

Main results

Theorem 3.1: *All directed complexes are homotopy polygraphs.*

Proof.

By definition. □

Indeed, the *(diagrammatic)* (∞, ω)*-categories*[2] are exactly the higher structures for which all directed complexes are homotopy polygraphs.

[2] non-univalent

Now for real

Theorem 3.2: *All directed complexes with acyclic atoms*[3] *are homotopy polygraphs, with respect to the standard definition of* (∞, n)*-categories.*

Proof.

This is the main Corollary of my latest preprint

A strengthened (∞, n)*-categorical pasting theorem*,

extending Tim Campion's results in

An (∞, n)*-categorical pasting theorem*.

□

[3] a somewhat arbitrary restriction. . .

Thanks! Questions?